N.N. Bogolyubov, Jr.

A METHOD FOR STUDYING MODEL HAMILTONIANS

A Minimax Principle for Problems in Statistical Physics

$$\langle \alpha_f^+(t)\,\alpha_f(\tau)\rangle_\Gamma = \int_{-\infty}^{+\infty} J_{\alpha_f^+,\,\alpha_f}(\omega)\, e^{i\omega(t-\tau)}\, d\omega,$$

$$\langle \alpha_f(\tau)\,\alpha_f^+(t)\rangle_\Gamma = \int_{-\infty}^{+\infty} J_{\alpha_f^+,\,\alpha_f}(\omega)\, e^{\omega/\theta}\, e^{i\omega(t-\tau)}\, d\omega,$$

$$\langle R_f^+(t)\,R_f(\tau)\rangle_\Gamma = \int_{-\infty}^{+\infty} J_{R_f^+,\,R_f}(\omega)\, e^{i\omega(t-\tau)}\, d\omega,$$

$$\langle R_f(\tau)\,R_f^+(t)\rangle_\Gamma = \int_{-\infty}^{+\infty} J_{R_f^+,\,R_f}(\omega)\, e^{\omega/\theta}\, e^{i\omega(t-\tau)}\, d\omega$$

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A METHOD FOR STUDYING MODEL HAMILTONIANS

A MINIMAX PRINCIPLE FOR PROBLEMS IN STATISTICAL PHYSICS

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SERIES EDITOR'S PREFACE

THERE is at present a growing interest in exact results in statistical mechanics and I am, therefore, very pleased to incorporate in the International Series of Monographs in Natural Philosophy the present volume by Dr. Bogolyubov in which he presents a detailed study of the use of model Hamiltonians which allow one to obtain solutions which are exact in the thermodynamic limit as the volume becomes infinite. The author has himself made many contributions to this field and his account bears the mark of an authority; it will, no doubt, be welcomed by workers in the field of statistical physics.

Oxford D. TER HAAR

PREFACE

IN THIS book methods are proposed for solving certain problems in statistical physics which contain four-fermion interaction.

It has been possible, by means of "approximating (trial) Hamiltonians", to distinguish a whole class of exactly soluble model systems.

An *essential difference* between the two types of problem with positive and negative four-fermion interaction is discovered and examined. The determination of exact solutions for the free energies, single-time and many-time correlation functions, T-products and Green's functions is treated for each type of problem.

The more general problem for which the Hamiltonian contains some terms with positive and others with negative four-fermion interaction is also investigated. On the basis of analysing and generalizing the results of Chapters 1 to 3, it becomes possible to formulate and develop a new principle, the *minimax principle*, for problems in statistical physics (Chapter 4).

I take this opportunity to express my gratitude to Dr. P. J. Shepherd for his comments and thorough checking of the manuscript. I also express my sincere thanks to Academician N. N. Bogolyubov for valuable remarks and to Dr. D. ter Haar for useful discussions.

The material of this book was the subject of theoretical seminars and lectures at Moscow State University. I consider it my pleasant duty to thank Professors A. N. Tavkhelidze (Corresponding Member of the USSR Academy of Sciences), A. A. Sokolov, I. M. Ternov, I. L. Bazarov and N. A. Chernikov, Dr. V. D. Kukin, and also the participants in the theoretical seminars at the V. A. Steklov Mathematics Institute and the Theoretical Physics Laboratory of Joint

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Moscow N. N. BOGOLYUBOV, Jr.

INTRODUCTION

§ 1. GENERAL REMARKS

In many-body theory most problems of physical interest are rather complicated and usually insoluble. Model systems permitting a mathematical treatment of these problems are therefore acquiring considerable interest.

Unfortunately, however, in concrete problems in many-body theory there is usually no adequate correspondence between a real system and its mathematical model; one must be content with a model whose properties differ substantially from those of the real system and in solving problems one must use approximate methods lacking the necessary mathematical rigour.

Of considerable interest in this connection is the study of those few models which have some resemblance to real systems yet admit exact solution; fundamental properties of many-body systems can be established in this way. Systems of non-interacting particles can be taken as examples of systems which can be solved exactly. Although, of course, this model seems rather trivial, it is used as a starting point in most problems in many-body theory. In the theory of metals one can often leave the mutual interaction of the valence electrons out of consideration. In the shell model of nuclei in its simplest form one can explain many general properties of nuclear spectra without introducing interaction between the particles into the treatment.

One of the most important problems in statistical physics is the study of exactly soluble cases. The fact is that this study makes an essential contribution to our understanding of the extremely complex problems of statistical physics and, in particular, serves as a basis for the approximate methods used in this field.

Up to the present time, the class of exactly soluble dynamical model systems has consisted mainly of one- and two-dimensional systems.†

In this study we shall concentrate on the treatment of certain model systems of a general type which can be solved exactly, e.g. models with four-fermion pair interaction which have as their origin the BCS model problems and are applicable in the theory of superconductivity; the determination of asymptotically exact solutions for these models has been investigated by Bogolyubov, Zubarev and Tserkovnikov (1957, 1960; see also Bogolyubov, Tolmachev and Shirkov, 1958).

In these papers an approximation procedure was formulated in which the ideas of a method based on the introduction of "approximating (trial) Hamiltonians"‡ were propounded and reasons were given for believing that the solution obtained was asymptotically exact on passing to the usual statistical mechanical limit $V \to \infty$.

Bogolyubov, Zubarev and Tserkovnikov (1957, 1960) considered the case of a model with the Hamiltonian

$$\begin{gathered} H = H_0 + H_{\text{int}}, \\ H_0 = \sum_{p,\,s} (E(p) - \mu) a^+_{ps} a_{ps}, \qquad (1) \\ H_{\text{int}} = -\sum_{p,\,p'} \frac{1}{V} \mathcal{J}(p, p') a^+_{-p,\,-1/2}\, a^+_{p,\,+1/2}\, a_{p',\,+1/2}\, a_{-p',\,-1/2}, \end{gathered}$$

where $a_{p,\,\pm 1/2}$ and $a^+_{p,\,\pm 1/2}$ are Fermi operators and V is the volume of the system. The kernel $\mathcal{J}(p, p')$ is assumed to be a real bounded function which effectively vanishes outside a certain region of momenta. The summation in H_{int} is performed over the momenta p and

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† See Lee and Yang, 1952; Yang and Yang, 1966; Lieb and Liniger, 1963; McGuire, 1964; Lieb, 1963; Yang, 1967, 1968, 1970; and Sutherland, 1968.

‡ Although the term "approximating Hamiltonian" is the literal translation of the Russian and is the form found in most of the English translations of the author's and other Russian articles, the term "trial Hamiltonian", which accords better with British and American usage, will be used in this book. *(Translator.)*

p' belonging to the energy layer:

$$E_F - \omega < E(p) < E_F + \omega.$$

It was shown that for such a Hamiltonian it is possible to construct the free energy asymptotically exactly (as $V \to \infty$). The idea of this method consisted in the introduction of a so-called "trial Hamiltonian" $H_0(C)$, which is a quadratic form of Fermi operators and depends on certain arbitrary constants C.

Such a Hamiltonian is easily diagonalized and it is then not difficult to calculate the free energy corresponding to it. In this work reasons were given for assuming that the free energy $F_0(C)$ becomes equal to F as $V \to \infty$. This result was obtained by means of perturbation theory. The derivation was based on the fact that in the perturbation theory series each of the terms used to calculate the correction to this solution is asymptotically small when $V \to \infty$. However, the question of the convergence of the perturbation theory series was not investigated and a more rigorous basis was required for the results of this work. In view of this, Bogolyubov, Zubarev and Tserkovnikov (1960) treated the problem without applying perturbation theory.

The system studied was of the BCS type and had the form

$$H = \sum_f T(f) a_f^+ a_f - \frac{1}{2V} \sum_{f,f'} \mathcal{J}(f,f') a_f^+ a_{-f}^+ a_{-f'} a_{f'} + \nu \mathcal{U}, \tag{2}$$

where

$$\mathcal{U} = \tfrac{1}{2} \sum_f w(f) \left(a_{-f} a_f + a_f^+ a_{-f}^+\right); \qquad \nu \geqslant 0.$$

$f = (\boldsymbol{p}, \sigma)$, $-f = (-\boldsymbol{p}, -\sigma)$, σ is the spin index, taking the values $\frac{1}{2}$ and $-\frac{1}{2}$, $\boldsymbol{p}$ is the momentum, which takes the usual series of quasi-discrete values $p_\alpha = 2\pi n_\alpha / L$, where, for a fixed value of $L (L^3 = V)$, n_α runs over all integers, $T(f) = (p^2/2m) - \mu$, μ is the chemical potential, and a_f and a_f^+ are operators satisfying the commutation relations of Fermi–Dirac statistics. $\mathcal{J}(f, f')$ and $w(f)$ are real functions with the properties

$$\mathcal{J}(f, f') = \mathcal{J}(f', f) = -\mathcal{J}(-f, f'),$$
$$w(-f) = -w(f).$$

For example,

$$\mathcal{J}(f,f') = \tfrac{1}{2}\mathcal{J}(\boldsymbol{p},\boldsymbol{p}')\,[\delta(\sigma-\sigma')-\delta(\sigma+\sigma')],$$
$$\mathcal{J}(\boldsymbol{p},\boldsymbol{p}') = \mathcal{J}(\boldsymbol{p}',\boldsymbol{p}) = \mathcal{J}(-\boldsymbol{p},\boldsymbol{p}'),$$

where $\delta(\sigma-\sigma')$ is the Kronecker symbol.

In this work a chain of coupled equations for the Green's functions was studied and it was shown that the Green's function for the "integrable problem" with Hamiltonian H_0 satisfies the whole chain of equations for the exact Hamiltonian (2) with an error of order $(1/V)$. From a purely mathematical point of view, however, it is obvious that reasonings of this type are not entirely final. Nevertheless, these papers are a substantial contribution to the study and determination of the asymptotically exact solution.

We remark that a rigorous proof of the asymptotic exactness of the results of Bogolyubov, Zubarev and Tserkovnikov (1957, 1960) presents substantial mathematical difficulties.

Such a problem was studied by Bogolyubov (1960 a, b) in a purely mathematical formulation in the case of zero temperature. He treated the model system characterized by the Hamiltonian (2) with the assumption that the kernel $\mathcal{J}(f,f')$ has the simple factorizable form:

$$\mathcal{J}(f,f') = \lambda(f)\,\lambda(f'). \tag{3}$$[†]

For the supplementary conditions it was assumed that the functions $\lambda(f)$ and $T(f)$ satisfy the following general conditions:

[†] From a physical point of view, it should be noted that the simple factorization of the kernel (3) describes so-called *s*-state interactions; when one needs to take account of higher states (e.g. *p*- and *d*-waves), the kernel must be written in the form

$$\mathcal{J}(f,f') = \sum_{\alpha=1}^{s} g_\alpha \lambda_\alpha(f)\,\lambda_\alpha(f'), \qquad g_\alpha > 0.$$

Some model systems of this type have been treated in the zero-temperature case, i.e. for the ground state of the system; this treatment, however, succeeded only in determining the ground-state energy asymptotically exactly as $V \to \infty$ (Tareeva, 1963, 1964).

(1) The functions $\lambda(f)$ and $T(f)$ are real and have the symmetry properties:

$$\lambda(-f) = -\lambda(f), \quad T(-f) = T(f),$$

(2)
$$\left.\begin{array}{c} |\lambda(f)| \leqslant \text{const} \\ T(f) \to \infty \end{array}\right\} \quad \text{for} \quad |f| \to \infty, \tag{4}$$

(3)
$$\frac{1}{V}\sum_{f} |\lambda_f| \leqslant \text{const} \quad \text{for} \quad V \to \infty,$$

(4)
$$\lim_{(V\to\infty)} \frac{1}{2V}\sum_{f} \frac{\lambda^2(f)}{\sqrt{\lambda^2(f)x + T^2(f)}} > 1,$$

for sufficiently small positive x.

This model problem (2) with the kernel (3) and the conditions (4) was investigated fully for zero temperature.

It was found that this problem can be solved asymptotically exactly as the volume of the system tends to infinity, i.e. the ground-state energy and the Green's functions and correlation functions characterizing the dynamical behaviour of the system can be determined.

We remark that in statistical physics we are always interested in the limiting values (as $V \to \infty$) of the dynamical quantities and functions being studied. In this work, Bogolyubov (1960 a, b) also showed, for application to the above case, how quasi-averages should be introduced; namely, source-terms were added to the Hamiltonian (2), proportional to pair-creation operators and pair-annihilation operators. After passing to the limit $V \to \infty$, the intensity of the sources was set to zero and a definition of quasi-averages was thereby constructed.

Of considerable interest is the analogous problem for arbitrary temperatures, i.e. $\theta \neq 0$ or $\theta = 0$. In this case, however, the most simple generalization of this problem is found to be impossible. The technique applied in the investigation of this problem was inherently limited to treating the ground state. Papers (Bogolyubov, Jr., 1965, 1966a) have been devoted to this complicated problem for model Hamiltonians of the form (2) (with attraction) in the case of non-zero temperatures.

In these papers a detailed study was made of a generalized model system (with attraction close to the Fermi surface), characterized by the Hamiltonian

$$H = T - 2V \sum_{1 \leq \alpha \leq s} g_\alpha J_\alpha J_\alpha^+ . \tag{5}$$

This system goes over to the usual BCS model system studied in the theory of superconductivity, if we take as the operators T and J_α the quadratic Fermi-operator forms

$$T = \sum_f T(f) a_f^+ a_f, \quad J_\alpha = \frac{1}{2V} \sum_f \lambda_\alpha(f) a_f^+ a_{-f}^+ . \tag{6}$$

Then for the Hamiltonian (5) the kernel $\mathcal{J}(f, f')$ will be

$$\mathcal{J}(f, f') = \sum_{1 \leq \alpha \leq s} g_\alpha \lambda_\alpha(f) \lambda_\alpha(f'). \tag{7}$$

In the above papers the case with all $g_\alpha = 1$ was studied. It is clear that with the appropriate choice of functions $\lambda_\alpha(f)$ we could obtain the systems studied by Tareeva (1963, 1964) at zero temperatures, and also more general systems† in which "pairs" interact not only in s-states, but also in p-, d-states, etc.

For the model problem (5) under consideration, we have constructed a trial Hamiltonian

$$H^0 = T - 2V \sum_{1 \leq \alpha \leq s} g_\alpha (C_\alpha J_\alpha^+ + C_\alpha^* J_\alpha) + 2V \sum_{1 \leq \alpha \leq s} g_\alpha |C_\alpha|^2 . \tag{8}$$

The complex constants $C_\alpha (1 \leq \alpha \leq s)$ occurring here are determined from the condition for the absolute minimum of the function

$$f_{H^0}(C_1, \ldots, C_s) = -\frac{\theta}{V} \ln \operatorname{Tr} e^{-H^0/\theta} \tag{9}$$

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† See Pitaevskiĭ, 1959; Privorotskiĭ, 1963; Anderson and Morel, 1960, 1961; Brueckner and Gammel, 1958; Anderson, 1962; Balian, Nosanow and Wertheimer, 1962; Brueckner, Soda, Anderson and Morel, 1960; Cooper, Mills and Sessler, 1959; Klein, 1962; Markowitz and Kadanoff, 1963; Thouless, 1960; Gor'kov and Galitskiĭ, 1961; Yang, 1962; and Bogolyubov, Jr., 1966b.

in the region of complex quantities $(C_1, \ldots, C_s)$. On the basis of the results of papers by the author (Bogolyubov, Jr., 1965, 1966a), we have developed a technique by means of which, with sufficiently general conditions imposed on the operators T and J_α, it is possible to find an upper bound for $\Delta = f_{H^0} - f_H$, the difference in the free energies per unit volume. It follows from this bound that this difference tends to zero as $V \to \infty$. The results obtained for finite s have been generalized further to the case when $s = \infty$. The results found were valid for both cases, $\theta > 0$ and $\theta = 0$.

As we have shown, for such systems it is possible to construct or find an expression for the free energy, which, since the technique has been developed for arbitrary temperatures, can be used in the treatment of phase transitions.

Furthermore, using the example of the model system (5) with $s = 1$, we (Bogolyubov, Jr., 1966a) have indicated a way of obtaining asymptotically exact estimates for the simplest, binary, correlation functions.

In a series of papers (Bogolyubov, Jr., 1966c, 1967, 1968 a, b, c, d, e, f, g, 1969, 1970 a, b, c, d, e, f), which form the basis of this book, we have developed a new method which permits us to study the systems (5) with kernels of the type (7) at any temperature, in cases when the parameters g_α are positive, negative or of different signs. We have succeeded in deriving asymptotically exact expressions not only for the free energies but also for the many-time correlation functions and Green's functions which completely characterize the dynamical behaviour of the system.

In the process we have made clear the necessity of introducing a new definition of quasi-averages, since the definition accepted until now has proved to be inadequate in the cases we have considered.

In order to unify all the model problems we have studied and to simplify appreciably our account of this material, the plan of the book has been arranged as follows.

All the model problems being treated are reduced to a general form, characterized by the Hamiltonian

$$\Gamma = \Gamma_a + H_1 \tag{10}$$

where

$$\Gamma_a = T - \frac{1}{2} \sum_f \{\Lambda^*(f) a_{-f} a_f + \Lambda(f) a_f^+ a_{-f}^+\} + \text{const}, \tag{11}$$

$$H_1 = -V \sum_\alpha G_\alpha (J_\alpha - C_\alpha)(J_\alpha^+ - C_\alpha^*), \tag{12}$$

$$T = \sum_f T(f) a_f^+ a_f, \quad J_\alpha = \frac{1}{2V} \sum_f \lambda_\alpha(f) a_f^+ a_{-f}^+, \tag{13}$$

and the function $\Lambda(f)$ satisfies the conditions I referred to below. The notation is the same as in the treatment of the model system (2); a_f and a_f^+ are Fermi operators, V is the volume of the system, and the summation over f is performed over the quasi-discrete set ϕ_V. In the sums over α, the index α takes integer values.

This model system is treated under certain supplementary conditions (conditions I) (Chapter 1, § 1), of which the main one is

$$\left\langle \sum_\alpha |G_\alpha| (J_\alpha - C_\alpha)(J_\alpha^+ - C_\alpha^*) \right\rangle_\Gamma \leqslant \varepsilon_V, \tag{14}$$

where $\varepsilon_V \to 0$ as $V \to \infty$. The symbol $\langle \ldots \rangle_\Gamma$ implies statistical averaging over the Hamiltonian Γ. These conditions are chosen in such a way that they are satisfied by all the problems considered later, in Chapters 3 and 4.

We note that the reduction of Hamiltonians to the form (10)–(13) and the proof that condition (14) is fulfilled require considerable mathematical resources.

For the technique of Chapters 1 and 2 to be applicable, it is necessary only that the conditions I be satisfied; then the results of Chapters 1 and 2 and the Theorems I–IV which are proved there are automatically extended to the cases of model systems with negative four-fermion interaction (cf. Chapter 3), model systems with positive four-fermion interaction, and also to the mixed case when the Hamiltonian contains both "attraction" and "repulsion" terms (cf. Chapter 4).

As has been pointed out, all the dynamical problems studied in this work can be reduced to the model Hamiltonian (10)–(13). This

Hamiltonian consists of two parts, one (11) quadratic in the Fermi operators, and the other (12) quartic in the Fermi operators.

We note that the quadratic form (11), which will be referred to below as the trial Hamiltonian, can be easily diagonalized by means of the well-known u–v transformation techniques, and for this system it turns out to be not difficult to calculate explicitly the corresponding expressions for the free energy, correlation functions, etc.

The quartic form (12), which describes the interaction, is in some sense, because of condition (14), a small correction.

It should be emphasized that an extremely complicated mathematical technique was required to study the "effect" of this correction on the free energy, correlation functions and Green's functions, and to show, further, that this effect disappears in the passage to the limit $V \to \infty$.

The main feature of Chapters 3 and 4 is the reduction to the "universal form" (10)–(13) of the model Hamiltonians dealt with in these chapters and an actual proof of condition (14) for the specific systems considered there.

In Chapter 1 we examine earlier methods (Bogolyubov, Jr., 1967) and the difficulties associated with the introduction of quasi-averages for certain particular model systems with four-fermion negative interaction. In this connection we formulate a new principle for treating model systems of the form (10)–(13) under sufficiently general conditions (cf. Chapter 1, § 1). The model systems (10)–(13) represent a fairly broad class of model systems, including the following important particular cases: (a) systems with four-fermion negative interaction (e.g. the BCS model systems studied in the theory of superconductivity), (b) systems with positive four-fermion interaction, and also (c) the mixed case for which the Hamiltonian contains both terms with "attraction" and terms with "repulsive" interaction.

For this general class of model systems we demonstrate how single-time and many-time correlation averages can be determined asymptotically exactly (as $V \to \infty$). In passing we point out the possibility of obtaining upper bounds which are uniform in θ.

In Chapter 2, using the results and upper bounds obtained in

Chapter 1, we prove generalized asymptotic relations for many-time correlation averages (composed of products of field functions), T-products and Green's functions.

Noting that the difference $n_f - n_{-f}$ in the numbers of particles with opposite momenta is an integral of motion for the systems (10)–(13), we formulate here special selection rules for averages composed of products of Fermi creation and annihilation operators.

On the basis of this treatment we formulate Theorem III and also the Theorem IV for the existence of generalized limits for averages taken over the trial Hamiltonian.

In Chapter 3 we examine a concrete application of the results of the first two chapters to model systems with four-fermion negative interaction. We draw attention here to certain difficulties associated with the definition of quasi-averages for model systems.

We develop a new method of introducing the auxiliary system and defining quasi-averages for a broad class of model systems containing negative interaction.

In §§ 1–3, we consider the question of the asymptotic expression for the free energy as $V \to \infty$.

A model system, characterized by the Hamiltonian (5), is studied. we treat the case when the operators T and J_α have the form (6), and all the $g_\alpha = 1$.

In § 1 of Chapter 3 we study certain properties of the free energies derived on the basis of the model Hamiltonian (5) and the trial Hamiltonian (8) (with all $g_\alpha = 1$). The results concerning the closeness of the free energies constructed on the basis of the Hamiltonians (5) and (8) are briefly formulated. The bounds obtained are found to be uniform in the temperature θ. We note, furthermore, that the existence of an asymptotic expression for the free energy constructed on the basis of the Hamiltonian (5) does not follow from the inequalities for the difference in the free energies constructed on the basis of the model Hamiltonian (5) and the trial Hamiltonian (8).

In this connection, in Chapter 3 we formulate those conditions for which we can give a correct proof of the existence of a limit as the volume tends to infinity for the free energy constructed on the basis of the model system (5), i.e.

$$\lim_{(V\to\infty)} f(H).$$

In § 7 of Chapter 3 we consider the question of the definition of the quasi-averages. Let $\mathcal{U}$ be some operator of the form for which the limit theorems are formulated in Chapters 1 and 2, e.g. a product of Fermi amplitudes or field functions.

It has been proposed that the quasi-average $\prec\mathcal{U}\succ$ of such an operator for the above Hamiltonian be defined as the limit

$$\prec\mathcal{U}_H\succ = \lim_{(\nu\to 0)}\left(\lim_{(V\to\infty)}\langle\mathcal{U}\rangle_\Gamma\right) \tag{15}$$

of the usual averages $\langle\mathcal{U}\rangle_\Gamma$ taken over the Hamiltonian Γ obtained from (5) by the addition of "source-terms"

$$\Gamma = H - V\sum_\alpha (\nu_\alpha J_\alpha^+ + \nu_\alpha^* J_\alpha). \tag{16}$$

In § 7 of Chapter 3 we draw attention to the difficulties associated with the definition (15). In this definition there is, for example, no indication of the region in which the parameters ν must lie or of how they must tend to zero in order to ensure convergence in the definition (15). It is shown in § 7 of Chapter 3 that, even in the simplest cases of binary averages for a kernel of the form (3), if ν tends to zero arbitrarily, the limit in the expression (15) may not exist, since this expression is a finite statistical average multiplied by the factor $\nu/|\nu|$; the presence of this factor means that as ν tends to zero the expression (15) does not tend to any limit. The limit does exist, for example, when ν tends to zero in such a way that the ratio $\nu/|\nu|$ remains constant. In the general case of a non-factorizing kernel (i.e. when the number of terms $s > 1$), with the passage to the limit $\nu \to 0$ the situation is found to be even more complicated. Apart from gauge invariance (due to the gauge group), other groups of transformations, e.g. the rotation group, may also be manifested.

To avoid this type of difficulty, in § 8 of Chapter 3 we propose that ν be taken proportional to $\tilde{C}$ with a constant coefficient of proportionality. Accordingly, it is shown in § 8 that the auxiliary Hamiltonian with sources takes the form

$$\Gamma = H + 2V \sum_{\alpha} g_\alpha \tau_\alpha (J_\alpha - \bar{C}_\alpha)(J_\alpha^+ - \bar{C}_\alpha^*), \tag{17}$$

in which the τ_α are positive parameters ($\alpha = 1, 2, \ldots, s$), and H is given by the expression (5). We emphasize that $\bar{C}_\alpha$ denotes the point at which the absolute minimum is attained of the function (9) taken after the passage to the limit $V \to \infty$, so that $\bar{C}_\alpha$ does not depend on V. We see that with the above choice (17) of the auxiliary Hamiltonian, no difficulties arise in defining the quasi-averages. These results are summarized by the following theorem:

THEOREM 3.IV. *If the conditions of Theorem* 3.III *are fulfilled and the Hamiltonian is represented by expression* (17) *in which*

$$0 < \tau_\alpha < 1 \qquad (\alpha = 1, 2, \ldots, s), \tag{18}$$

then the following inequalities are valid:

$$0 \leqslant f_V(\Gamma) - f_V(H) \leqslant \delta_V + \delta_V \to 0 \quad \text{as} \quad V \to \infty, \tag{19}$$

$$\sum_{\alpha} g_\alpha \langle (J_\alpha - \bar{C}_\alpha)(J_\alpha^+ - \bar{C}_\alpha^*) \rangle_\Gamma \leqslant \frac{\delta_V + \delta_V}{2\tau_0} \to 0 \quad \text{as} \quad V \to \infty, \tag{20}$$

where τ_0 *is the smallest of the quantities* $\tau_1, \ldots, \tau_s$.

On the other hand,

$$\Gamma = H(\bar{C}) - 2V \sum_{\alpha} g_\alpha (1 - \tau_\alpha)(J_\alpha - \bar{C}_\alpha)(J_\alpha^+ - \bar{C}_\alpha^*), \tag{21}$$

where $H(\bar{C})$ is the trial Hamiltonian, defined by expression (8). By virtue of the above result for the smallness of the "correction term" (20) quartic in the Fermi operators, all the conditions I of Chapter 1 will be fulfilled and consequently all the results of Chapters 1 and 2 are automatically extended to the above case.

To summarize, we shall write down theorems on the existence of limits for the quasi-averages

$$\prec \mathcal{U} \succ_H = \lim_{(V \to \infty)} \langle \mathcal{U} \rangle_\Gamma = \lim_{(V \to \infty)} \langle \mathcal{U} \rangle_{H(\bar{C})}, \tag{22}$$

in which τ can take any values from the range (18).

We draw attention to a surprising feature of the asymptotic relations (22). In spite of the fact that the Hamiltonian (21) depends on the parameters τ_α, the expression in the left-hand side of the asymptotic relations (22), which is constructed on the basis of this Hamiltonian, is found to be independent of the parameters τ_α lying in the range (18).

Using the new definition (22) of the quasi-averages and taking into account that $H(\bar{C})$ is independent of the parameters τ_α lying in the range (18), we can write the old definition in the form

$$\lim_{(\tau \to 0)} \left(\lim_{(V \to \infty)} \langle \mathcal{U} \rangle_\Gamma \right) = \lim_{(V \to \infty)} \langle \mathcal{U} \rangle_{H(\bar{C})}. \tag{23}$$

The expression in the round brackets in the left-hand side of (23) always has as its limit the right-hand side of (23), provided that the parameters τ_α lie in the range (18). Consequently, when all the τ_α tend to zero while remaining positive, the limit as $\tau \to 0$ is trivial.

The main argument of § 7 of Chapter 3 is the proof of the inequality (20), which is based on the inequality (19). It is not difficult to see that the asymptotic relation for the free energies (as $V \to \infty$) also follows from the inequality (20).

We remark that for our method it is not necessary to confine ourselves to the case $g_\alpha > 0$. In Chapter 4 we examine the situation when the constants g_α have different signs, and in this case it is also possible to reduce the Hamiltonian to the form treated in Chapters 1 and 2.

In Chapter 4 we consider model systems with positive and negative interaction components. In § 1 we investigate the properties of the free energy for a system with positive four-fermion interaction. We prove the existence of an absolute maximum for the free energy constructed on the basis of the corresponding trial Hamiltonian.

It is shown that the solutions of the equations (formulae (4.9)) for the point $\bar{C}$ at which the absolute maximum of the free-energy function occurs are unique.

In § 2 we examine the question of whether the analogue of Theorem 3.I is valid for the case of a model system with positive interaction components when the operators T and J have arbitrary unspecified form.

It is shown that, in the general case when the operator forms of T and J (for interaction of the repulsive type) are unspecified but satisfy the same general conditions as for Theorem 3.I, the free energy constructed on the basis of the model Hamiltonian does not tend, as $V \rightarrow \infty$, to the free energy constructed on the basis of the corresponding trial Hamiltonian.

In § 3 of Chapter 4 we investigate the problem of calculating the free energy and correlation averages for a model system with positive four-fermion interaction, corresponding to repulsion of fermions. The essential difference between this and the case of systems with negative four-fermion interaction (cf. Chapter 3) is that the free energy constructed on the basis of a trial Hamiltonian regarded as a function of the variables $\ldots C_\alpha \ldots$ has an absolute maximum for a certain fixed set of quantities $\ldots \bar{C}_\alpha \ldots$, and the maximum point is unique. As a result we obtain upper bounds for the free energy and quasi-averages and from these there follows an unexpected result: the contribution to the Hamiltonian of the system from a four-fermion positive pair interaction is found to be negligibly small for large systems ($V \rightarrow \infty$) (cf. § 3).

On the basis of this treatment, in § 4 we also investigate a model system of a more general form (4.30). This study is essentially auxiliary for § 6, in which we formulate the minimax principle. As a result, we obtain bounds for the free energies and quasi-averages.

In § 5 we prove that the solutions of the equations (4.47) are unique and also show that these solutions $\bar{C}(U)$ are continuous functions and possess continuous partial derivatives with respect to the variables $\ldots U_\beta \ldots$ and $\ldots U_\beta^* \ldots$. In this proof we make use of a theorem concerning implicit functions (since all the conditions for its applicability are fulfilled). The results obtained are summarized by Theorem 4.1.

In § 6 we investigate Hamiltonians with coupling constants of different signs and formulate the minimax principle. We construct an auxiliary system and set up the asymptotic relations for the free energies and quasi-averages. An expression is found for the asymptotically exact calculation of the free energy using the minimax principle (formula (4.73)).

We emphasize that in deriving a proof for the determination of an asymptotically exact solution for a model system (5) in which the g_α take different signs, we start from a specially constructed trial Hamiltonian $H(C, S)$ which is a quadratic form in the Fermi operators and depends on the groups of complex variables $C_1, \ldots, C_m$ and $S_1, \ldots, S_r$. These variables, which appear initially as independent of each other, are then related by means of the minimax technique, namely, from the condition for the absolute maximum of the free energy with respect to the variables $C_1, \ldots, C_m$ for fixed $S_1, \ldots, S_r$ we obtain a system of equations from which a unique solution $C = C(S)$ can be determined. The solution found is then put into the trial Hamiltonian. Using this trial Hamiltonian, we construct the free energy and in the final upper-bound inequalities take the absolute minimum

$$\min_{(S)} f\{H(C(S), S)\}$$

of the free energy with respect to the variables $S_1, \ldots, S_r$.

It is found that taking the maximum and minimum in the reverse order leads to an incorrect result.

We also consider the question of the definition of the quasi-averages and the construction of a Hamiltonian with sources. As a result we obtain upper bounds, uniform with respect to the temperature, for the corresponding quasi-averages. It is shown that this problem reduces to the Hamiltonian (10)–(13) and, since all the supplementary conditions I are fulfilled, we can apply the limit theorems of Chapters 1 and 2 to this case and obtain the same results for the quasi-averages as in Chapter 3.

In this book we have solved the complicated problem of constructing upper bounds for many-time averages for the case of a fairly broad class of model systems with four-fermion interaction. The methods proposed in this book for solving this problem will undoubtedly find application not only for the model systems, associated with the theory of superconductivity, considered here. The theoretical methods developed in Chapters 1 and 2 are applicable to a much broader class of model systems from statistical physics and the theory of elementary particles.

§ 2. REMARKS ON QUASI-AVERAGES[†]

The expression "quasi-average" will be used frequently in this book; we therefore include this introductory section to clarify, without the required mathematical rigour, the physical meaning of the concept of a "quasi-average" and of the connection between quasi-averages and the question of the degeneracy of a state of statistical equilibrium. This section is of an introductory nature and has no direct connection with the main text of Chapters 1–4.

Here, using a number of simple model examples borrowed from statistical physics, we shall elucidate the physical meaning of the quasi-averages introduced in the papers of Bogolyubov (1960 a, b).

First we shall examine the idea of the degeneracy of a state of statistical equilibrium. We remark that the concept of degeneracy is well known in quantum mechanics (Heisenberg, 1966).

In treating problems on the determination of eigenfunctions in quantum mechanics, we know that the usual form of perturbation theory, developed for problems without degeneracy, cannot be applied to problems with degeneracy; for such problems it must be modified.

Because of the existence of additive conservation laws, there are always cases of degeneracy in problems in statistical physics.

It might seem at first sight, however, that in these problems the degeneracy has no effect and need not be taken into account in practice. In fact, in problems in quantum mechanics, a linear manifold of eigenfunctions can correspond to one energy eigenvalue; in this case, the eigenfunctions contain undetermined constants.

[†] See Bogolyubov, 1960b, 1963. (See also *Selected Papers of N. N. Bogolyubov*, Vol. 3, p. 174, Naukovaya Dumka, Kiev, 1971 (in Russian).)

In statistical physics, however, the average value of any dynamical variable $\mathcal{U}$ is always uniquely defined by

$$\langle\mathcal{U}\rangle = \frac{\operatorname{Tr} \mathcal{U}\, e^{-H/\theta}}{\operatorname{Tr} e^{-H/\theta}}.$$

Consequently, the Green's functions and T-products constructed from the usual averages must also be uniquely defined.

It may appear from this that in studying a state of statistical equilibrium, e.g. by means of a diagram technique, we need not take the presence of degeneracy into account. In fact, the situation is more complicated.

To form an intuitive idea of the nature of the difficulties arising here, we shall consider the case of an ideal isotropic ferromagnet.

We shall take the Heisenberg model system

$$H = -\tfrac{1}{2} \sum_{f_1, f_2} I(f_1 - f_2)(S_{f_1} \cdot S_{f_2}), \tag{24}$$

where the f are spatial points corresponding to the lattice sites (lying in the volume V), S_f is the spin operator of an electron at site f, and $I(f_1 - f_2)$ is the exchange integral. For definiteness we put $I(f_1 - f_2) \geqslant 0$ for all f_1 and f_2.

We note that, for the system (24), each component of the total spin

$$S^\alpha = \sum_f S_f^\alpha \qquad (\alpha = x, y, z) \tag{25}$$

is an integral of motion.

We have also

$$\begin{gathered} S^x S^y - S^y S^x = iS^z, \quad S^y S^z - S^z S^y = iS^x, \\ S^z S^x - S^x S^z = iS^y. \end{gathered} \tag{26}$$

Taking these commutation relations into account, we write

$$i \operatorname{Tr} (S^z e^{-H/\theta}) = \operatorname{Tr} \{(S^x S^y - S^y S^x)\, e^{-H/\theta}\}.$$

But since S^x commutes with H, we obtain

$$\mathrm{Tr}\,(S^y S^x\, e^{-H/\theta}) = \mathrm{Tr}\,(S^y\, e^{-H/\theta} S_x) = \mathrm{Tr}\,(S^x S^y\, e^{-H/\theta})$$

and, therefore,

$$\mathrm{Tr}\,(S^z\, e^{-H/\theta}) = 0.$$

Repeating these arguments for the components S^y and S^x, we find

$$\mathrm{Tr}\,(S^\alpha\, e^{-H/\theta}) = 0, \qquad (\alpha = x, y, z). \tag{27}$$

We introduce the magnetization vector referred to unit volume:

$$\mathcal{M}^\alpha = \mu \frac{1}{V} \sum_f S_f^\alpha = \mu \frac{1}{V} S^\alpha.$$

We have

$$\mathrm{Tr}\,(\mathcal{M}^\alpha\, e^{-H/\theta}) = 0,$$

and, consequently,

$$\langle \mathcal{M}^\alpha \rangle = \lim_{(V \to \infty)} \frac{\mathrm{Tr}\, \mathcal{M}^\alpha\, e^{-H/\theta}}{\mathrm{Tr}\, e^{-H/\theta}} = 0, \qquad (\alpha = x, y, z), \tag{28}$$

i.e. the mean magnetic moment of the system is zero, which corresponds to the isotropy of this dynamical system with respect to the spin-rotation group. We emphasize that the expression (28) is valid for all temperatures θ, and in particular for temperatures below the Curie point. It is known, however, that at temperatures below the Curie point the magnetic moment of the system is non-zero, although its direction can be chosen arbitrarily. In this sense, a state of statistical equilibrium in this system is degenerate.

We now switch on an external magnetic field $\nu \boldsymbol{e}$ ($\nu > 0$, $\boldsymbol{e}^2 = 1$), thus replacing the Hamiltonian (24) by

$$H_{\nu e} = H + \nu(\boldsymbol{e} \cdot \mathcal{M})\, V. \tag{29}$$

Bearing in mind a characteristic property of isotropic ferromagnets at temperatures below the Curie point, we see that

$$\langle \mathcal{M}^\alpha \rangle = e^\alpha M_\nu, \qquad (\alpha = x, y, z), \tag{30}$$

where M_ν will tend to a finite (non-zero) limit when the intensity ν of the external magnetic field tends to zero.

In this case we thus have an "instability" in the usual averages; on addition of the term $\nu(e\cdot\mathcal{M})V$ with infinitesimally small ν[†] to the Hamiltonian (24), the average $\langle\mathcal{M}^\alpha\rangle$ experiences a finite non-zero increment:

$$em; \qquad m = \lim_{\nu\to 0} M_\nu .$$

For the system with the Hamiltonian (24) we introduce the concept of a "quasi-average".

We take the dynamical variable $\mathcal{A}$ to be a linear combination of the products

$$\mathcal{A} = S_{f_1}^{\alpha_1}(t_1)\ldots S_{f_r}^{\alpha_r}(t_r). \tag{31}$$

We shall define the quasi-average $\prec\mathcal{A}\succ$ of this quantity as

$$\prec\mathcal{A}\succ = \lim_{(\nu\to 0)} \langle\mathcal{A}\rangle_{\nu e},$$

where $\langle\mathcal{A}\rangle_{\nu e}$ is the usual average of $\mathcal{A}$ for the Hamiltonian $H_{\nu e}$.

Thus, the presence of degeneracy is directly reflected in the quasi-averages by their dependence on the arbitrary unit vector e.

It is not difficult to see that the quasi-average is related to the usual average $\langle\mathcal{A}\rangle$ by

$$\langle\mathcal{A}\rangle = \int \prec\mathcal{A}\succ de. \tag{32}$$

It is clear that quasi-averages are more convenient for the description of the above case of degeneracy of a state of statistical equilibrium and also more "physical" than the usual averages. The latter are the quasi-averages averaged over all directions e.

The usual averages

$$\langle S_{f_1}^{\alpha_1}(t_1)\ldots S_{f_r}^{\alpha_r}(t_r)\rangle \tag{33}$$

must be invariant with respect to the spin-rotation group.

[†] When speaking of infinitesimally small ν, we mean that ν tends to zero after the statistical mechanical passage to the limit $V \to \infty$.

The corresponding quasi-averages

$$\prec S_{f_1}^{\alpha_1}(t_1) \dots S_{f_r}^{\alpha_r}(t_r) \succ \tag{34}$$

will only possess the property of covariance; on rotation of the spins, the vector $\boldsymbol{e}$ must be subjected to the same rotation, so that the expression (34) is unchanged. Thus, the selection rules for the usual averages, resulting from the invariance of the latter with respect to the spin-rotation group, will not apply to the quasi-averages. It is seen that the unit vector $\boldsymbol{e}$, i.e. the direction of the magnetization vector, characterizes the degeneracy of this state of statistical equilibrium. To remove the degeneracy, the direction of $\boldsymbol{e}$ must be fixed. We shall take the z-axis for this direction. In this case, all the quasi-averages become well-defined numbers. It is with these averages that the theory of ferromagnetism is concerned. In other words, to remove the degeneracy of a state of statistical equilibrium with respect to the spin-rotation group, we must include in the Hamiltonian H an invariant extra term $\nu \mathcal{M}_z V$ with infinitesimally small ν. In the above example of an isotropic ferromagnet, the perturbation term is the energy of interaction of the system with an external magnetic field.

We turn now to consider a second example containing degeneracy, taken from the theory of the crystalline state. We shall take a dynamical system of spinless particles with two-particle interaction, characterized by a Hamiltonian in the second-quantized representation

$$\begin{aligned} H = {} & \sum_{p} \left(\frac{p^2}{2m} - \lambda \right) a_p^+ a_p \\ & + \frac{1}{2V} \sum_{p_1,\, p_2,\, p_1',\, p_2'} a_{p_1}^+ a_{p_2}^+ a_{p_2'} a_{p_1'} v(\boldsymbol{p}_1 - \boldsymbol{p}_1')\, \delta(\boldsymbol{p}_1 + \boldsymbol{p}_2 + \boldsymbol{p}_1' - \boldsymbol{p}_2'), \end{aligned} \tag{35}$$

in which $\delta(\boldsymbol{p})$ is the Kronecker δ and $v(\boldsymbol{p})$ is the Fourier transform of the potential energy of interaction $\Phi(\boldsymbol{r})$ of a pair of particles. We shall further assume that this interaction is such that our dynamical system is crystalline at fairly low temperatures $\theta < \theta_{cr}$.

We shall consider the particle-number density $\varrho(\boldsymbol{r})$, which must clearly be a periodic function of $\boldsymbol{r}$ with the period of the crystal lattice.

It would seem natural to assume that $\varrho(\boldsymbol{r})$ is equal to the usual average value of the density operator $\Psi^{+}(\boldsymbol{r})\,\Psi(\boldsymbol{r})$:

$$\varrho(\boldsymbol{r}) = \langle \Psi^{+}(\boldsymbol{r})\,\Psi(\boldsymbol{r})\rangle = \frac{1}{V}\sum_{q}\sum_{k}\langle a_k^{+}a_{k+q}\rangle\, e^{i(\boldsymbol{q}\cdot\boldsymbol{r})}. \tag{36}$$

This, however, is not so.

In fact, in this case a conservation law for the total momentum is valid:

$$\boldsymbol{K} = \sum_{k}\boldsymbol{k}\, a_k^{+}a_k, \qquad H\boldsymbol{K}-\boldsymbol{K}H = 0.$$

Taking into account the translational invariance of averages of the form

$$\langle \mathfrak{U}\rangle = \langle \Psi^{+}(t, \boldsymbol{r}+\boldsymbol{\xi})\,\Psi(t, \boldsymbol{r}+\boldsymbol{\xi})\rangle, \tag{37}$$

where

$$\Psi(t, \boldsymbol{r}) = \frac{1}{\sqrt{V}}\sum_{k} a_k(t)\, e^{i(\boldsymbol{k}\cdot\boldsymbol{r})}, \tag{38}$$

it is not difficult to arrive at the selection rules:

$$\langle a_k^{+}a_{k+q}\rangle = 0 \quad \text{if} \quad q \neq 0,$$

whence

$$\langle \Psi^{+}(\boldsymbol{r})\,\Psi(\boldsymbol{r})\rangle = \frac{1}{V}\sum_{k}\langle a_k^{+}a_k\rangle = \frac{\mathfrak{N}}{V} = \text{const}. \tag{39}$$

Thus, the usual average of the density operator cannot be equal to the periodic function $\varrho(\boldsymbol{r})$. It is clear that this position has come about because of the degeneracy of the state of statistical equilibrium under consideration.

A crystalline lattice can, as a whole, be positioned arbitrarily in space. In particular, our Hamiltonian possesses translational invariance and, therefore, the lattice can always be given an arbitrary translation. There is nothing to distinguish any particular spatial position of the crystal lattice, and when we take the usual average, we average at the same time over all possible positions of the lattice. To remove the degeneracy, we include in the Hamiltonian (35) a term of the form

$$\nu \int U(\boldsymbol{r})\,\Psi^{+}(\boldsymbol{r})\,\Psi(\boldsymbol{r})\,d\boldsymbol{r}; \qquad \nu > 0, \quad \nu \to 0 \tag{40}$$

corresponding to an infinitesimal external field $\nu U(\boldsymbol{r})$. For $U(\boldsymbol{r})$ we take a periodic function of $\boldsymbol{r}$ with the periodicity of the lattice, so that the external field νU removes the degeneracy, fixing the position of the crystal in space.

Since we naturally consider only physical, stable systems, it is clear that the inclusion of an infinitesimal external field can only produce an infinitesimal change in the physical properties of the system being studied.

By taking the usual average of the density operator $\Psi^+(\boldsymbol{r})\,\Psi(\boldsymbol{r})$ for the Hamiltonian H_ν with infinitesimal ν, we in fact obtain an average for the system with the original Hamiltonian H, but without the additional averaging over the spatial positions of the crystal lattice (as a whole), since its position is now fixed.

In this way we obtain the observed spatial particle-distribution density $\varrho(\boldsymbol{r})$. We shall define the quasi-averages formally by

$$\begin{aligned}\prec \ldots \Psi^+(t_j, \boldsymbol{r}_j) \ldots \Psi(t_s, \boldsymbol{r}_s) \succ \\ = \lim_{(\nu \to 0)} \langle \ldots \Psi^+(t_j, \boldsymbol{r}_j) \ldots \Psi(t_s, \boldsymbol{r}_s) \ldots \rangle_{H_\nu}.\end{aligned}$$

Then, as we have already remarked,

$$\prec \Psi^+(\boldsymbol{r})\,\Psi(\boldsymbol{r}) \succ = \varrho(\boldsymbol{r}).$$

Bearing in mind that

$$\prec \Psi^+(\boldsymbol{r})\,\Psi(\boldsymbol{r}) \succ = \frac{1}{V}\sum_q \left\{ \sum_k \prec a_k^+ a_{k+q} \succ \right\} e^{i(\boldsymbol{q}\cdot\boldsymbol{r})}$$

we see that the quasi-averages

$$\langle a_k^+ a_{k'} \rangle \qquad (k' \neq k) \tag{41}$$

cannot all be equal to zero.

It is clear that the selection rules arising from the law of conservation of the total momentum are not obeyed by the quasi-averages we have introduced.

We note that the quasi-averages depend in general on a number of arbitrary parameters, e.g. on the arbitrary vector $\boldsymbol{\xi}$. In fact, by replacing $U(\boldsymbol{r})$ by the permissible function $U(\boldsymbol{r}+\boldsymbol{\xi})$, it is not difficult to see that the quasi-averages (41) are replaced by

$$\prec a_k^+ a_{k'} \succ e^{i(k-k'\cdot\xi)}.$$

The quasi-averages become uniquely defined when the function $U(\boldsymbol{r})$ is fixed.

We have considered cases of degeneracy of a state of statistical equilibrium associated with a conservation law for the total spin vector and the total momentum vector. In both examples, the degeneracy could be removed and quasi-averages introduced by including a suitable infinitesimal external field.

We give one further example, when the degeneracy is associated with a conservation law for the total number of particles. We take the well-known example of the Bose–Einstein condensation of an ideal gas. We write the Hamiltonian of the system in the form

$$H = -\lambda a_0^+ a_0 + \sum_{|k|>\varepsilon} \left(\frac{k^2}{2m} - \lambda\right) a_k^+ a_k, \qquad \varepsilon > 0. \tag{42}$$

The degeneracy will be lifted if we add the infinitesimal term $\nu(a_0 + a_0^+)\sqrt{V}$.

In the above examples, the degeneracy was associated with the existence of additive conservation laws, or, in other words, with the presence of invariance with respect to the corresponding groups of transformations. We remark that not all the conservation laws applying in a given system bring about degeneracy. In the last example (the Bose gas), the degeneracy was associated only with the law of conservation of the number of particles. In the corresponding quasi-averages, only those selection rules which resulted from this law were violated. In the second example (the crystal lattice), the degeneracy was associated only with the law of conservation of momentum. Selection rules due to, for example, the particle-number conservation law were not violated. One could extend the list of examples by considering cases of degeneracy associated with other groups of transformations, or with several groups of transformations simultaneously.

We shall not dwell on this here. We shall pass to more general considerations and introduce the appropriate definitions.

We shall take a certain macroscopic system with Hamiltonian H. To H we add some infinitesimal terms corresponding to the external fields or sources violating the additive conservation laws and thereby obtain another Hamiltonian H_ν $(\nu > 0)$. Then, if all the averages

$$\langle A \rangle, \quad A = \ldots \Psi^+(t_j, \boldsymbol{r}_j) \ldots \Psi(t_s, \boldsymbol{r}_s) \ldots \tag{43}$$

receive only infinitesimal increments, the state of statistical equilibrium will be said to be non-degenerate. Conversely, if some of the averages (43) receive finite increments in going from H to the infinitesimally close Hamiltonian H_ν, the state of statistical equilibrium will be said to be degenerate. We confine ourselves to treating stable systems, since only they have physical meaning.

For cases of degeneracy, it is appropriate to introduce quasi-averages in place of the usual averages:

$$\prec A \succ = \lim_{(\nu \to 0)} \langle A \rangle_{H_\nu}.$$

As we have already seen from the above examples the, selection rules resulting from the additive conservation laws are not necessarily all obeyed for the quasi-averages. We emphasize that by defining quasi-averages we are constructing a two-limit technique: first we put the volume equal to infinity, and then we put ν equal to zero.

CHAPTER 1

PROOF OF THE ASYMPTOTIC RELATIONS FOR THE MANY-TIME CORRELATION FUNCTIONS

§ 1. GENERAL TREATMENT OF THE PROBLEM. SOME PRELIMINARY RESULTS AND FORMULATION OF THE PROBLEM

We begin here with the problem of the asymptotic calculation of quasi-averages and prove a number of preliminary theorems.

For a natural approach to the proper formulations of these we recall a series of results which we established earlier for a Hamiltonian of the form

$$H = T - 2VgJJ^{+} \tag{1.1}$$

in which

$$T = \sum_{f} T(f)\, a_f^{+} a_f; \quad T(f) = \frac{p^2}{2m} - \mu,$$

$$J = \frac{1}{2V} \sum_{f} \lambda(f)\, a_f^{+} a_{-f}^{+} \tag{1.2}$$

a_f and a_f^{+} are Fermi amplitudes, m, μ and g are positive constants, and

$$f = (\boldsymbol{p}, \sigma)$$

where σ is the spin index, which takes the values $\pm\frac{1}{2}$. The function

$$\lambda(f) = \lambda(\boldsymbol{p}, \sigma)$$

occurring in (1.2) is real and continuous in the spherical layer

$$\left|\frac{p^2}{2m}-\mu\right| \leqslant \Lambda$$

(where Λ is a certain positive constant) and equal to zero outside it; it also possesses the property of antisymmetry

$$\lambda(-f) = -\lambda(f); \quad -f = (-\boldsymbol{p}, -\sigma).$$

We note finally that in the summation "over f" the components p_α ($\alpha = 1, 2, 3$) of the vector $\boldsymbol{p}$ take the values $2\pi n_\alpha/L$, while n_α runs over all integers ($-\infty$, ∞). $L^3 = V$, where V is the volume of the system, which below will tend to ∞.

To introduce the quasi-averages we add to the Hamiltonian H terms with "pair sources", e.g.

$$-\nu V(J+J^+)$$

where ν is a positive constant.

Thus, the Hamiltonian under consideration will be

$$\Gamma = T-2VgJJ^+-\nu V(J+J^+). \tag{1.3}$$

The quasi-averages for the Hamiltonian H are introduced as the limits, as $V \to \infty$, of the usual averages for the Hamiltonian Γ, with the *sequential* passage to the limit

$$\prec\ldots\succ_H = \lim_{\substack{\nu\to 0\\(\nu>0)}} \lim_{V\to\infty} \langle\ldots\rangle_\Gamma.$$

We have shown (Bogolyubov, Jr., 1967)[†] that the simplest binary-type correlation averages

$$\langle a_f^+(t)\, a_f(\tau)\rangle_\Gamma, \quad \langle a_f^+(t)\, a_{-f}^\pm(\tau)\rangle_\Gamma, \quad \langle a_{-f}(t)\, a_f(\tau)\rangle_\Gamma$$

are asymptotically ($V \to \infty$) close to the corresponding averages taken for the "trial Hamiltonian":[†]

[†] We note that this technique has also been found to be useful in the study of exactly soluble quasi-spin models (Bogolyubov, Jr., 1968h; Thirring and Wehrl, 1967; and Thirring, 1968).

$$\Gamma_a(C) = T-2Vg(CJ^+ + C^*J - C^*C) - \nu V(J+J^+)$$
$$= T-2Vg\left\{\left(C+\frac{\nu}{2g}\right)J^+ + \left(C^*+\frac{\nu}{2g}\right)J\right\}+2VgC^*C.$$

(There is, however, another approach, viz. to deal with infinite volume from the start. Such a situation was studied in a paper by Petrina (1970).) The quantity C occurring in $\Gamma_a(C)$ is determined from the condition that the free energy has an absolute minimum

$$f\{\Gamma_a(C)\} = \min$$

in the whole complex C-plane.

Since $\Gamma_a(C)$ is a quadratic form in the Fermi amplitudes, this Hamiltonian can be diagonalized by means of a u–v transformation:

$$a_f = u(f)\alpha_f - v(f)\alpha_f^+ , \tag{1.4}$$

where α_f and α_f^+ are new Fermi amplitudes, and

$$u(f) = \frac{1}{\sqrt{2}}\sqrt{1+\frac{T(f)}{E(f)}},$$

$$V(f) = -\frac{\lambda(f)\left(C+\frac{\nu}{2g}\right)}{\sqrt{2}\left|\lambda(f)\left(C+\frac{\nu}{2g}\right)\right|}\sqrt{1-\frac{T(f)}{E(f)}},$$

$$E(f) = \sqrt{T^2(f)+4\lambda^2(f)g^2\left|C+\frac{\nu}{2g}\right|^2}.$$

In the new Fermi amplitudes the Hamiltonian will take the form

$$\Gamma_a(C) = \left\{2gC^*C - \frac{1}{2V}\sum_f (E(f)-T(f))\right\}V + \sum_f E(f)\alpha_f^+\alpha_f \tag{1.5}$$

[†] In this formula we should have written $2VgC^*C\cdot\hat{1}$, where $\hat{1}$ is the unit operator; however, since this will not lead to misunderstandings anywhere, below we shall not write out the unit operator explicitly.

so that the free energy per unit volume calculated on the basis of this Hamiltonian will be

$$\begin{aligned} f\{\Gamma_a(C)\} &= 2gC^*C - \frac{1}{2V}\sum_f \{E(f) - T(f)\} \\ &\quad - \frac{\theta}{V}\sum_f \ln\left(1 + e^{-E(f)/\theta}\right). \end{aligned} \tag{1.6}$$

It is clear from this that the absolute minimum of $f\{\Gamma_a(C)\}$ as a function of the complex variable is found at a real value of C, and at a value such that

$$C + \frac{\nu}{2g} > 0.$$

This minimizing value of C depends in general on V and ν:

$$C = C(V, \nu).$$

In our above-mentioned paper (Bogolyubov, Jr., 1967) it was shown that

$$C(V, \nu) \to C(\nu)$$

as

$$V \to \infty \qquad (\nu\,(\text{fixed}) > 0) \tag{1.7}$$

and

$$C(\nu) \xrightarrow[(\nu > 0)]{\nu \to 0} C(0).$$

Here $C = C(\nu)$ realizes the absolute minimum of the asymptotic expression

$$\begin{aligned} f_\infty\{\Gamma_a(C)\} &= \lim_{V \to \infty} f\{\Gamma_a(C)\} \\ &= 2gC^*C - \frac{1}{2(2\pi)^3}\int df\{E(f) - T(f)\} \\ &\quad - \frac{\theta}{(2\pi)^3}\int df \ln\left(1 + e^{-E(f)/\theta}\right). \end{aligned}$$

As throughout the book, the "integral over f" denotes integration over p and summation over σ:

$$\int df(\ldots) = \sum_\sigma \int dp(\ldots).$$

The value $C(0) \geqslant 0$ is chosen as the number giving the absolute minimum of the function

$$f_\infty\{H_a(C)\} = 2gC^2 - \frac{1}{2(2\pi)^3}\int df\{E(f)-T(f)\}$$
$$-\frac{\theta}{(2\pi)^3}\int df \ln\left(1+e^{-E(f)/\theta}\right),$$

in which

$$E(f) = \sqrt{T^2(f)+4g^2C^2\lambda^2(f)}.$$

As we have noted already, we have proved in our work that the difference of the binary averages constructed on the basis of the model (Γ) and trial (Γ_a) Hamiltonians tends to zero as $V \to \infty$ for any fixed value of $\nu > 0$.

On the other hand, in view of (1.4) and (1.5), these averages are calculated easily for Γ_a.

For example, we have

$$\langle a_f^+(t)\, a_f(\tau)\rangle_{\Gamma_a} = u^2(f)\, e^{iE(f)(t-\tau}\, \frac{e^{-E(f)/\theta}}{1+e^{-E(f)/\theta}}$$
$$+\, v^2(f)\, e^{-iE(f)(t-\tau)}\, \frac{1}{1+e^{-E(f)/\theta}}\,.$$
$$\langle a_f^+(t)\, a_{-f}^+(\tau)\rangle_{\Gamma_a} = u(f)\, v(f) \left\{ e^{iE(f)(t-\tau)}\, \frac{e^{-E(f)/\theta}}{1+e^{-E(f)/\theta}} \right.$$
$$\left. -\, e^{-iE(f)(t-\tau)}\, \frac{1}{1+e^{-E(f)/\theta}} \right\}. \tag{1.8}$$

As can be seen, the right-hand sides here are defined for *all* p, and not only for the quasi-discrete values

$$p = \left(\frac{2\pi n_1}{L},\ \frac{2\pi n_2}{L},\ \frac{2\pi n_3}{L}\right),$$

which are the only values for which the amplitudes a_f and a_f^+, and thereby the left-hand sides of the expressions (1.8), are defined.

Moreover, the right-hand sides of (1.8) as functions of f depend on V only through the quantity $C = C(V, \nu)$.

The passage to the limit $V \to \infty$ therefore reduces, because of (1.7), to replacing $C(V, \nu)$ by $C(\nu)$ in these functions. The subsequent passage to the limit $\nu \to 0$ $(\nu > 0)$ corresponds to replacement of $C(\nu)$ by $C(0)$.

Thus, if we put $\nu = 0$, $C = C(0)$ in the functions of f occurring in the right-hand sides of (1.8), these functions will represent the corresponding quasi-averages for the Hamiltonian H.

We note that the most complicated of our proofs was that establishing the relations

$$\langle \ldots \rangle_{\Gamma} - \langle \ldots \rangle_{\Gamma_a} \to 0 \tag{1.9}$$

as

$$V \to \infty$$

for binary expressions of the type indicated above. To prove them, we had to show first that

$$\langle (J - C(V, \nu))(J^{+} - C(V, \nu)) \rangle_{\Gamma} \to 0 \tag{1.10}$$

as

$$V \to \infty$$

and then establish the asymptotic relations (1.9a). We must emphasize that the problem of investigating situations with more complicated averages was not solved in the papers cited (Bogolyubov, Jr., 1966 a, c, 1967).

Moreover, the proof of the properties (1.10) and (1.7) was based on specific features of the Hamiltonian (1.1) and could not be extended to model Hamiltonians of more general form.

In our work (Bogolyubov, Jr., 1966 a, c, 1967, 1968e), we have constructed a new method which enables us to extend the above-mentioned results to the case of many-time averages of Fermi amplitudes or field functions, and, furthermore, for model Hamiltonians of more complicated structure.

If we wished to apply this method to the investigation of the Hamiltonian Γ it would be appropriate to start from the representation:

$$\begin{aligned} \Gamma &= \Gamma_a(C(\nu)) - 2gV(J - C(\nu))(J^{+} - C(\nu)) \\ &= T - \tfrac{1}{2} \sum_f \Lambda(f) \{a_f^{+} a_{-f}^{\pm} + a_{-f} a_f\} + 2gVC^2 \\ &\quad - 2gV(J - C(\nu))(J^{+} - C(\nu)), \end{aligned} \tag{1.11}$$

where

$$\Lambda(f) = 2g\lambda(f)\left(C(\nu)+\frac{\nu}{2g}\right). \tag{1.12}$$

In this case, in view of (1.7) and (1.10)

$$\langle (J-C(\nu))\,(J^{+}-C(\nu))\rangle_{\Gamma} \leqslant \varepsilon_V \to 0. \tag{1.13}$$

However, we shall not study only this Hamiltonian here.

As will be shown under much wider conditions in the following chapters, model Hamiltonians with properly chosen source terms can also be reduced to a form similar to (1.11).

We turn now to consider the situation when

$$\Gamma = \Gamma_a + H_1, \tag{1.14}$$

where

$$\Gamma_a = \sum_f T(f)\,a_f^{+}a_f - \tfrac{1}{2}\sum_f \{\Lambda^{*}(f)\,a_{-f}a_f + \Lambda(f)\,a_f^{+}a_{-f}^{+}\} + \mathcal{K},$$

where

$$\mathcal{K} = \text{const}, \qquad T(f) = \frac{p^2}{2m} - \mu,$$

and

$$H_1 = -V\sum_{\alpha} G_{\alpha}(J_{\alpha}-C_{\alpha})\,(J_{\alpha}^{+}-C_{\alpha}^{*}), \tag{1.15}$$

where

$$J_{\alpha} = \frac{1}{2V}\sum_f \lambda_{\alpha}(f)a_f^{+}a_{-f}^{+}.$$

Here the summation over f runs over the above-mentioned quasi-discrete set, which we shall call the set ϕ_V. We shall examine this model system under the following conditions, which we shall call conditions I.

1. The functions $\lambda_{\alpha}(f)$ and $\Lambda(f)$ are defined and bounded in the whole space ϕ of the points $f = (\boldsymbol{p}, \sigma)$.
2. The series

$$\sum_{\alpha} |G_{\alpha}|\,|\lambda_{\alpha}(f)|^2 = P(f),$$

converges uniformly in ϕ and the function $P(f)$ represented by it satisfies the inequalities:

$$P(f) \leqslant M_1 = \text{const},$$

$$\frac{1}{V}\sum_f P(f) \leqslant M_2 = \text{const}.$$

3. The inequality

$$\left\langle \sum_\alpha |G_\alpha| (J_\alpha - C_\alpha)(J_\alpha^+ - C_\alpha^*) \right\rangle_\Gamma \leqslant \varepsilon_V$$

is satisfied, where

$$\varepsilon_V \to 0 \quad \text{as} \quad V \to \infty.$$

4. The function $\Lambda(f)$ and the constants C_α satisfy the inequalities

$$\frac{1}{V}\sum_f |\Lambda(f)|^2 \leqslant M_\Lambda,$$

$$\sum_\alpha |G_\alpha||C_\alpha|^2 \leqslant M_C.$$

5. The functions $\lambda_\alpha(f)$ and $\Lambda(f)$ are antisymmetric with respect to reflection:[†]

$$\lambda_\alpha(-f) = -\lambda_\alpha(f), \quad \Lambda(-f) = -\Lambda(f).$$

Here α takes integer values. (In general, the sums over α imply an infinite number of terms; in Chapters 3 and 4, however, we consider problems in which α takes a finite number of values.)

With these conditions we prove a series of theorems on the asymptotic closeness of averages taken over the Hamiltonians Γ and Γ_a respectively.

[†] This last condition is not, essentially, a restrictive one. Any sum of the form

$$\sum_f F(f)\, a_f^+ a_{-f}^+; \quad \sum_f F(f)\, a_{-f} a_f$$

can always be reduced to the form:

$$\sum_f \frac{F(f) - F(-f)}{2} a_f^+ a_{-f}^+; \quad \sum_f \frac{F(f) - F(-f)}{2} a_{-f} a_f,$$

in which the coefficient function $[F(f) - F(-f)]/2$ is already antisymmetric with respect to the reflection: $f \to -f$.

In the following chapters we shall reduce the Hamiltonians studied to the form (1.14), (1.15) and, as soon as we have established the validity of the conditions I, we can then make use of certain theorems. We turn now to the proof of these theorems.

§ 2. EQUATIONS OF MOTION AND AUXILIARY OPERATOR INEQUALITIES

We turn first of all to the equations of motion for the model Hamiltonian Γ. We have

$$\begin{aligned} i\frac{da_f}{dt} &= a_f\Gamma-\Gamma a_f = T(f)\,a_f-\Lambda(f)\,a^+_{-f} \\ &\quad -\sum_\alpha G_\alpha\lambda_\alpha(f)\,a^+_{-f}(J^+_\alpha-C^*_\alpha). \end{aligned} \tag{1.16}$$

Taking the conjugate and replacing f by $-f$, we find also

$$\begin{aligned} i\frac{da^+_{-f}}{dt} &= -T(f)\,a^+_{-f}-\Lambda^*(f)\,a_f \\ &\quad -\sum_\alpha G_\alpha\lambda^*_\alpha(f)\,(J_\alpha-C_\alpha)\,a_f. \end{aligned} \tag{1.17}$$

As can be seen, the first two terms in the right-hand sides of these equations originate from the so-called trial Hamiltonian Γ_a. But this Hamiltonian can be diagonalized by means of a canonical u–v transformation

$$\begin{aligned} a_f &= u(f)\,\alpha_f-v(f)\,\alpha^+_{-f} \\ a^+_{-f} &= u(f)\,\alpha^+_{-f}+v^*(f)\,\alpha_f \end{aligned} \tag{1.18}$$

in which we put

$$u(f)=\frac{1}{\sqrt{2}}\sqrt{1+\frac{T(f)}{E(f)}},\quad v(f)=-\frac{\Lambda(f)}{\sqrt{2}\,|\Lambda(f)|}\sqrt{1-\frac{T(f)}{E(f)}}, \tag{1.19}$$

$$E(f)=\sqrt{T^2(f)+|\Lambda(f)|^2}.$$

As a result, this Hamiltonian is reduced to the form

$$\Gamma_a=\sum_f E(f)\,\alpha^+_f\alpha_f+\text{const.}$$

Expressing the "new" Fermi amplitudes α_f in terms of the "old" a_f, we obtain, for example,

$$\alpha_f = u(f)\, a_f + v(f)\, a^+_{-f}.$$

Hence, taking eqns. (1.16) and (1.17) into account, we shall have

$$i\,\frac{d\alpha_f}{dt} = E(f)\,\alpha_f + R_f, \tag{1.20}$$

where

$$\begin{aligned} R_f &= -u(f)\, R_f^{(1)} - v(f)\, R_f^{(2)}, \\ R_f^{(1)} &= \sum_\alpha G_\alpha \lambda_\alpha(f)\, a^+_{-f}(J^+_\alpha - C^*_\alpha), \\ R_f^{(2)} &= \sum_\alpha G_\alpha \lambda^*_\alpha(f)\,(J_\alpha - C_\alpha)\, a_f. \end{aligned} \tag{1.21}$$

We must also keep in mind the conjugate equation

$$-i\,\frac{d\alpha^+_f}{dt} = E(f)\,\alpha^+_f + R^+_f, \tag{1.22}$$

in which

$$\begin{aligned} R^+_f &= -u(f)\, R_f^{+(1)} - v(f)\, R_f^{+(2)}, \\ R_f^{+(1)} &= \sum_\alpha G_\alpha \lambda^*_\alpha(f)\,(J_\alpha - C_\alpha)\, a_{-f}, \\ R_f^{+(2)} &= \sum_\alpha G_\alpha \lambda_\alpha(f)\, a^+_f(J^+_\alpha - C^*_\alpha). \end{aligned} \tag{1.23}$$

Since our aim is to compare different quantities for the Hamiltonians Γ and Γ_a, it is natural that somehow we shall have to estimate the asymptotic smallness of the "correction terms" R_f and R^+_f; for this it will be necessary first of all to establish a series of operator inequalities.

We shall show, in particular, that

$$\Big(\sum_j x_j A_j\Big)\Big(\sum_j x^*_j A^+_j\Big) \leqslant \Big(\sum_j |x_j|^2\Big)\Big(\sum_j A_j A^+_j\Big), \tag{1.24}$$

where the x_j are complex numbers and the A_j are operators. We have, in fact,

$$\Big(\sum_j x_j A_j\Big)\Big(\sum_j x_j^* A_j^+\Big) = \sum_{j,k} x_j x_k^* A_j A_k^+$$
$$= \tfrac{1}{2}\sum_{j,k}\{x_j x_k^* A_j A_k^+ + x_k x_j^* A_k A_j^+\}.$$

But

$$(x_k^* A_j - x_j^* A_k)(x_k A_j^+ - x_j A_k^+) \geqslant 0,$$

i.e.

$$x_k^* x_k A_j A_j^+ + x_j x_j^* A_k A_k^+ \geqslant x_k x_j^* A_k A_j^+ + x_j x_k^* A_j A_k^+ .$$

Consequently,

$$\Big(\sum_j x_j A_j\Big)\Big(\sum_j x_j^* A_j^+\Big) \leqslant \tfrac{1}{2}\sum_{j,k}\{|x_k|^2 A_j A_j^+ + |x_j|^2 A_k A_k^+\}$$
$$= \sum_k |x_k|^2 \sum_j A_j A_j^+,$$

which is what we wished to prove.

We shall now apply the inequality (1.24) to find a bound for $R_f^+ R_f$ defined using (1.21) and (1.23). We have

$$R_f^+ R_f \leqslant (u^2(f) + |v(f)|^2)(R_f^{+(1)} R_f^{(1)} + R_f^{+(2)} R_f^{(2)})$$
$$= R_f^{+(1)} R_f^{(1)} + R_f^{+(2)} R_f^{(2)}, \qquad (1.25)$$

and also

$$R_f R_f^+ \leqslant R_f^{(1)} R_f^{+(1)} + R_f^{(2)} R_f^{+(2)}. \qquad (1.26)$$

Then, putting

$$x_\alpha = \frac{G_\alpha}{\sqrt{|G_\alpha|}}\lambda_\alpha^*(f), \qquad A_f = \sqrt{|G_\alpha|}\,(J_\alpha - C_\alpha)\, a_f$$

into (1.24), we obtain

$$R_f^{+(1)} R_f^{(1)} \leqslant \Big(\sum_\alpha |G_\alpha|\,|\lambda_\alpha(f)|^2\Big)\Big(\sum_\alpha |G_\alpha|\,(J_\alpha - C_\alpha)\, a_{-f} a_{-f}^+ (J_\alpha^+ - C_\alpha^*)\Big).$$

But if we have an operator inequality

$$A \leqslant B,$$

the inequality

$$UAU^+ \leqslant UBU^+$$

is also true. Since

$$a_{-f}a^{+}_{-f} \leqslant 1,$$

we can write

$$(J_\alpha - C_\alpha)\, a_{-f}a^{+}_{-f}(J^{+}_\alpha - C^{*}_\alpha) \leqslant (J_\alpha - C_\alpha)(J^{+}_\alpha - C^{*}_\alpha).$$

Consequently,

$$R^{+(1)}_f R^{(1)}_f \leqslant P(f) \sum_\alpha |G_\alpha| (J_\alpha - C_\alpha)(J^{+}_\alpha - C^{*}_\alpha). \tag{1.27}$$

In an entirely analogous way, we find for $R^{(2)}_f$:

$$R^{+(2)}_f R^{(2)}_f \leqslant P(f) \sum_\alpha |G_\alpha|\, a^{+}_f (J^{+} - C^{*}_\alpha)(J_\alpha - C_\alpha) a_f .$$

With the intention of using the situation when a^{+}_f and a_f are found to be "squeezed" between $(J_\alpha - C_\alpha)$ and $(J^{+}_\alpha - C^{*}_\alpha)$, we shall have to interchange these factors, since a^{+}_f commutes with J_α and a_f with J^{+}_α. For this, we note that

$$\begin{aligned}
(J^{+}_\alpha - C^{*}_\alpha)(J_\alpha - C_\alpha) &= (J_\alpha - C_\alpha)(J^{+}_\alpha - C^{*}_\alpha) + J^{+}_\alpha J_\alpha - J_\alpha J^{+}_\alpha \\
&= (J_\alpha - C_\alpha)(J^{+}_\alpha - C^{*}_\alpha) + \frac{2}{4V^2} \sum_f |\lambda_\alpha(f)|^2 (a_f a^{+}_f - a^{+}_f a_f) \\
&\leqslant (J_\alpha - C_\alpha)(J^{+}_\alpha - C^{*}_\alpha) + \frac{1}{2V^2} \sum_f |\lambda(f)|^2 ,
\end{aligned}$$

since

$$\begin{aligned}
J^{+}_\alpha J_\alpha - J_\alpha J^{+}_\alpha &= \frac{1}{4V^2} \sum_f |\lambda_\alpha(f)|^2 (a_{-f}a^{+}_{-f} - a^{+}_{-f}a_{-f} + a_f a^{+}_f - a^{+}_f a_f) \\
&= \frac{1}{2V^2} \sum_f |\lambda_\alpha(f)|^2 (a_f a^{+}_f - a^{+}_f a_f) \leqslant \frac{1}{2V^2} \sum_f |\lambda_\alpha(f)|^2 .
\end{aligned}$$

Hence follows

$$\begin{aligned}
& a^{+}_f (J^{+}_\alpha - C^{*}_\alpha)(J_\alpha - C_\alpha) a_f \\
&\quad \leqslant a^{+}_f (J_\alpha - C_\alpha)(J^{+}_\alpha - C^{*}_\alpha) a_f + \frac{1}{2V^2} \sum_f |\lambda_\alpha(f)|^2 \\
&\quad = (J_\alpha - C_\alpha)\, a^{+}_f a_f (J^{+}_\alpha - C^{*}_\alpha) + \frac{1}{2V^2} \sum_f |\lambda_\alpha(f)|^2 \\
&\quad \leqslant (J_\alpha - C_\alpha)(J^{+}_\alpha - C^{*}_\alpha) + \frac{1}{2V^2} \sum_f |\lambda_\alpha(f)|^2 .
\end{aligned}$$

Thus,

$$R_f^{+(2)}R_f^{(2)} \leqslant P(f)\left\{\sum_\alpha |G_\alpha|\,(J_\alpha - C_\alpha)(J_\alpha^+ - C_\alpha^*) + \frac{1}{2V^2}\sum_f P(f)\right\}$$

$$\leqslant P(f)\left\{\sum_\alpha |G_\alpha|\,(J_\alpha - C_\alpha)(J_\alpha^+ - C_\alpha^*) + \frac{M_2}{2V}\right\}.$$

Therefore, taking (1.25) and (1.26) into account, we obtain

$$R_f^+R_f \leqslant 2P(f)\left\{\sum_\alpha |G_\alpha|\,(J_\alpha - C_\alpha)(J_\alpha^+ - C_\alpha^*) + \frac{M_2}{4V}\right\}. \tag{1.28}$$

We also obtain, entirely analogously,

$$R_fR_f^+ \leqslant 2P(f)\left\{\sum_\alpha |G_\alpha|\,(J_\alpha - C_\alpha)(J_\alpha^+ - C_\alpha^*) + \frac{M_2}{4V}\right\}. \tag{1.29}$$

Hence, on the basis of conditions I, we see, in particular, that

$$\begin{aligned} \langle R_f^+R_f\rangle_\Gamma &\leqslant 2P(f)\left\{\varepsilon_V + \frac{M_2}{4V}\right\} \leqslant 2M_1\left\{\varepsilon_V + \frac{M_2}{4V}\right\}, \\ \langle R_fR_f^+\rangle_\Gamma &\leqslant 2P(f)\left\{\varepsilon_V + \frac{M_2}{4V}\right\} \leqslant 2M_1\left\{\varepsilon_V + \frac{M_2}{4V}\right\}. \end{aligned} \tag{1.30}$$

§ 3. ADDITIONAL INEQUALITIES

Before we can begin to establish bounds for the differences of the averages taken over the Hamiltonians Γ and Γ_a, we shall have to obtain further inequalities for expressions of the form

$$\begin{aligned} &\langle \beta_{f_1} \ldots \beta_{f_s} R_f^+R_f\beta_{f_s}^+ \ldots \beta_{f_1}^+\rangle_\Gamma, \\ &\langle \beta_{f_1} \ldots \beta_{f_s} R_fR_f^+\beta_{f_s}^+ \ldots \beta_{f_1}^+\rangle_\Gamma, \end{aligned} \tag{1.31}$$

in which

$$\beta_{f_j} = \alpha_{f_j} \quad \text{or} \quad \overset{+}{\alpha}_{f_j}.$$

Because of (1.28) and (1.29), this problem reduces to the problem of obtaining bounds for

$$D_\alpha = \langle \beta_{f_1} \ldots \beta_{f_s}(J_\alpha - C_\alpha)(J_\alpha^+ - C_\alpha^*)\beta_{f_s}^+ \ldots \beta_{f_1}^+\rangle_\Gamma.$$

We shall move the operator $(J_\alpha - C_\alpha)$ successively to the left. We find

$$\begin{aligned} D_\alpha = \langle (J_\alpha - C_\alpha)\beta_{f_1} \dots \beta_{f_s}(J_\alpha^+ - C_\alpha^*)\beta_{f_s}^+ \dots \beta_{f_1}^+\rangle_\Gamma \\ + \sum_{j=1}^{s} \langle B_j(J_\alpha^+ - C_\alpha^*)\beta_{f_s}^+ \dots \beta_{f_1}^+\rangle_\Gamma, \end{aligned} \tag{1.32}$$

where the operator B_j is obtained if the operator β_{f_j} in the product

$$\beta_{f_1} \dots \beta_{f_s} \tag{1.33}$$

is replaced by

$$\beta_{f_j} J_\alpha - J_\alpha \beta_{f_j}. \tag{1.34}$$

To calculate this difference, we shall examine two separate cases:

$$\beta_{f_j} = \alpha_{f_j} \quad \text{and} \quad \beta_{f_j} = \alpha_{f_j}^+.$$

We have

$$\begin{aligned} \alpha_{f_j} J_\alpha - J_\alpha \alpha_{f_j} &= \frac{1}{2V} \sum_f \lambda_\alpha(f) \left\{ \alpha_{f_j} a_f^+ a_{-f}^+ - a_f^+ a_{-f}^+ \alpha_{f_j} \right\} \\ &= \frac{1}{2V} \sum_f \lambda_\alpha(f) \left\{ [\alpha_{f_j} a_f^+ + a_f^+ \alpha_{f_j}]\, a_{-f}^+ - a_f^+ [\alpha_{f_j} a_{-f}^+ + a_{-f}^+ \alpha_{f_j}] \right\}. \end{aligned} \tag{1.35}$$

But, in view of (1.18),

$$\begin{aligned} a_f^+ &= u(f)\,\alpha_f^+ - v^*(f)\,\alpha_{-f}, \\ a_{-f}^+ &= u(f)\,\alpha_{-f}^+ + v^*(f)\,\alpha_f. \end{aligned}$$

It is clear that in the sum (1.35) the only terms which remain are those in which $f = f_j$ or $-f = f_j$. We obtain

$$\alpha_{f_j} J_\alpha - J_\alpha \alpha_{f_j} = \frac{\lambda_\alpha(f_j)}{V} u(f_j)\, a_{-f_j}^+,$$

and also

$$\alpha_{f_j}^+ J_\alpha - J_\alpha \alpha_{f_j}^+ = -\frac{\lambda_\alpha(f_j)}{V} v^*(f_j)\, a_{f_j}^+.$$

We have, consequently, a bound for the norm† of expression (1.34):

$$|\beta_{f_j} J_\alpha - J_\alpha \beta_{f_j}| \leqslant \frac{|\lambda_\alpha(f_j)|}{V}.$$

---

† Everywhere below, $|\dots|$ denotes the usual norm of the relevant operators in the second-quantized representation in the Hilbert space $\mathcal{L}$.

Since the norms of all the operators β_f are not greater than unity, and B_j is obtained from the product (1.33) by replacing β_{f_j} by the expression (1.34), we see that

$$\langle B_j B_j^+\rangle_\Gamma \leqslant \frac{|\lambda_\alpha(f_j)|^2}{V^2}.$$

Making use now of the general inequality

$$|\langle \mathcal{B}\mathcal{V}\rangle_\Gamma| \leqslant \sqrt{\langle \mathcal{B}\mathcal{B}^+\rangle_\Gamma \langle \mathcal{V}^+\mathcal{V}\rangle_\Gamma}, \tag{1.36}$$

we apply it to formula (1.32). We obtain

$$\begin{aligned} D_\alpha &\leqslant \sqrt{\langle (J_\alpha - C_\alpha)\,\beta_{f_1} \ldots \beta_{f_s}\beta_{f_s}^+ \ldots \beta_{f_1}^+(J_\alpha^+ - C_\alpha^*)\rangle_\Gamma}\,\sqrt{D_\alpha} \\ &\quad + \sum_{j=1}^{s} \frac{|\lambda_\alpha(f_j)|}{V}\sqrt{D_\alpha} \\ &\leqslant \sqrt{\langle (J_\alpha - C_\alpha)\,(J_\alpha^+ - C_\alpha^*)\rangle_\Gamma}\;\sqrt{D_\alpha} + \sum_{j=1}^{s} \frac{|\lambda_\alpha(f_j)|}{V}\sqrt{D_\alpha}. \end{aligned}$$

For brevity we put

$$\langle (J_\alpha - C_\alpha)\,(J_\alpha^+ - C_\alpha^*)\rangle_\Gamma = \Delta_\alpha.$$

We write

$$\sqrt{D_\alpha} \leqslant \sqrt{\Delta_\alpha} + \sum_{j=1}^{s} \frac{|\lambda_\alpha(f_j)|}{V},$$

and

$$D_\alpha \leqslant \Delta_\alpha + 2\sqrt{\Delta_\alpha}\sum_{j=1}^{s} \frac{|\lambda_\alpha(f_j)|}{V} + \left(\sum_{j=1}^{s} \frac{|\lambda_\alpha(f_j)|}{V}\right)^2.$$

Applying this inequality to the averages (1.31) and taking (1.28) and (1.29) into account, we obtain

$$\begin{aligned} \langle \beta_{f_1} \ldots \beta_{f_s} R_f^+ R_f \beta_{f_s}^+ \ldots \beta_{f_1}^+\rangle_\Gamma &\leqslant 2P(f)\frac{M_2}{4V} + 2P(f)\Bigg\{\sum_\alpha |G_\alpha|\,\Delta_\alpha \\ &\quad + 2\sum_\alpha |G_\alpha|\sqrt{\Delta_\alpha}\sum_{j=1}^{s} \frac{|\lambda_\alpha(f_j)|}{V} + \sum_\alpha |G_\alpha|\left(\sum_{j=1}^{s} \frac{|\lambda_\alpha(f_j)|}{V}\right)^2\Bigg\} \\ &\leqslant 2P(f)\frac{M_2}{4V} + 2P(f)\Bigg\{\varepsilon_V + 2\sqrt{\varepsilon_V}\sqrt{\sum_\alpha |G_\alpha|\left(\sum_{j=1}^{s} \frac{|\lambda_\alpha(f_j)|}{V}\right)^2} \\ &\quad + \sum_\alpha |G_\alpha|\left(\sum_{j=1}^{s} \frac{|\lambda_\alpha(f_j)|}{V}\right)^2\Bigg\}. \end{aligned}$$

But

$$\left(\sum_{j=1}^{s} |\lambda_\alpha(f_j)|\right)^2 \leqslant s \sum_{j=1}^{s} |\lambda_\alpha(f_j)|^2.$$

Hence,

$$\sum_\alpha |G_\alpha| \left(\sum_{j=1}^{s} \frac{|\lambda_\alpha(f_j)|}{V}\right)^2 \leqslant \frac{s}{V^2} \sum_{j=1}^{s} P(f_j) \leqslant \frac{s^2}{V^2} M_1.$$

Thus,

$$\begin{aligned} &\langle \beta_{f_1} \ldots \beta_{f_s} R_f^+ R_f \beta_{f_s}^+ \ldots \beta_{f_1}^+ \rangle_\Gamma \\ &\quad \leqslant 2P(f) \left\{ \frac{M_2}{4V} + \left(\sqrt{\varepsilon_V} + \frac{s}{V} \sqrt{M_1} \right)^2 \right\} \\ &\quad \leqslant 2M_1 \left\{ \frac{M_2}{4V} + \left(\sqrt{\varepsilon_V} + \frac{s}{V} \sqrt{M_1} \right)^2 \right\}, \end{aligned} \tag{1.37}$$

and, analogously,

$$\begin{aligned} &\langle \beta_{f_1} \ldots \beta_{f_s} R_f R_f^+ \beta_{f_{1s}}^+ \ldots \beta_{f_s}^+ \rangle_\Gamma \\ &\quad \leqslant 2M_1 \left\{ \frac{M_2}{4V} + \left(\sqrt{\varepsilon_V} + \frac{s}{V} \sqrt{M_1} \right)^2 \right\}. \end{aligned} \tag{1.38}$$

Having established the inequalities (1.30), (1.37) and (1.38), we can now turn to the proof of the asymptotic closeness of averages for the Hamiltonians Γ_a and Γ.

§ 4. BOUNDS FOR THE DIFFERENCE OF THE SINGLE-TIME AVERAGES

First of all we study the question of "single-time averages" and obtain asymptotic estimates for differences of the form

$$\langle \beta_{f_1} \ldots \beta_{f_s} \rangle_\Gamma - \langle \beta_{f_1} \ldots \beta_{f_s} \rangle_{\Gamma_a}. \tag{1.39}$$

We remark that these differences need be examined only for *even* values of s. For odd s, both terms in (1.39) are identically zero. In fact, both the Hamiltonians Γ and Γ_a are invariant with respect to the canonical interchange of Fermi amplitudes:

$$a_f \to -a_f, \qquad a_f^+ \to -a_f^+,$$

i.e., in the amplitudes α,

$$\alpha_f \to -\alpha_f, \qquad \alpha^+ \to -\alpha_f^+.$$

Therefore,

$$\langle \beta_{f_1} \dots \beta_{f_s} \rangle_{\Gamma, \Gamma_a} = (-1)^s \langle \beta_{f_1} \dots \beta_{f_s} \rangle_{\Gamma, \Gamma_a},$$

from which our remark follows. In the determination of estimates for the above differences, the spectral representations[†] for two-time averages are of fundamental importance:

$$\begin{aligned} \langle A(t)\, B(\tau) \rangle_\Gamma &= \int_{-\infty}^{+\infty} J_{A,B}(\omega)\, e^{i\omega(t-\tau)}\, d\omega, \\ \langle B(\tau)\, A(t) \rangle_\Gamma &= \int_{-\infty}^{+\infty} J_{A,B}(\omega)\, e^{\omega/\theta} e^{i\omega(t-\tau)}\, d\omega, \end{aligned} \tag{1.40}$$

where $A(t)$ and $B(\tau)$ are certain operators in the Heisenberg representation, defined as functions of time by the equations of motion

$$i\frac{dA(t)}{dt} = A(t)H - HA(t); \qquad A(0) = A,$$

$$i\frac{dB(\tau)}{d\tau} = B(\tau)H - HB(\tau); \qquad B(0) = B.$$

We remark, incidentally, that we shall use these spectral representations only for a fixed volume V; in this case, the Hamiltonian Γ has a discrete spectrum. This is ensured by the conditions I.4 (cf. Chapter 1, § 1).

We denote the complete set of its eigenfunctions by φ_ν and the corresponding eigenvalues by E_ν.

Then, as one can convince oneself,

$$\langle A(t)\, B(\tau) \rangle_\Gamma = \frac{1}{Q} \sum_{\nu,\mu} (\varphi_\nu, A\varphi_\mu)(\varphi_\mu, B\varphi_\nu)\, e^{-(E_\nu/\theta) + i(E_\nu - E_\mu)(t-\tau)}, \tag{1.41}$$

where $Q = \mathrm{Tr}\, e^{-H/\theta}$, and

$$J_{A,B}(\omega) = \frac{1}{Q} \sum_{\nu,\mu} (\varphi_\nu, A\varphi_\mu)(\varphi_\mu, B\varphi_\nu)\, e^{-E_\nu/\theta}\, \delta(E_\nu - E_\mu - \omega).$$

[†] See Bogolyubov and Tyablikov, 1959; Bonch-Bruevich, 1956 a, b; Zubarev, 1960; and Bogolyubov, Jr. and Sadovnikov, 1962, 1963.

Consequently, the integrals occurring in (1.40) are found to be discrete sums.

It is not the detailed structure of the so-called spectral intensity function $J_{A,B}(\omega)$ which will be of special interest to us, but only a number of properties, which we shall use constantly, of such spectral functions. We have in mind the following general relations:

$$\begin{gathered} J_{A,A^+}(\omega) \geqslant 0, \\ \int_{-\infty}^{+\infty} J_{A,A^+}(\omega)\, d\omega = \langle AA^+\rangle_\Gamma, \\ \int_{-\infty}^{+\infty} J_{A,A^+}(\omega)\, e^{\omega/\theta}\, d\omega = \langle A^+A\rangle_\Gamma, \end{gathered} \tag{1.42}$$

$$\begin{aligned} &\int_{(l)} |h_1(\omega)\, h_2(\omega)\, J_{A,B}(\omega)|\, d\omega \\ &\quad \leqslant \sqrt{\int_{(l)} |h_1(\omega)|^2 J_{A,A^+}(\omega)\, d\omega \int_{(l)} |h_2(\omega)|^2 J_{B^+,B}(\omega)\, d\omega} \end{aligned}$$

for the continuous functions $h_1(\omega)$ and $h_2(\omega)$, where (l) denotes any particular infinite integral.

We shall give special attention to the spectral representations

$$\begin{aligned} \langle \alpha_f^+(t)\, \alpha_f(\tau)\rangle_\Gamma &= \int_{-\infty}^{+\infty} J_{\alpha_f^+,\alpha_f}(\omega)\, e^{i\omega(t-\tau)}\, d\omega, \\ \langle \alpha_f(\tau)\, \alpha_f^+(t)\rangle_\Gamma &= \int_{-\infty}^{+\infty} J_{\alpha_f^+,\alpha_f}(\omega)\, e^{\omega/\theta}\, e^{i\omega(t-\tau)}\, d\omega, \end{aligned} \tag{1.43}$$

$$\begin{aligned} \langle R_f^+(t)\, R_f(\tau)\rangle_\Gamma &= \int_{-\infty}^{+\infty} J_{R_f^+,R_f}(\omega)\, e^{i\omega(t-\tau)}\, d\omega, \\ \langle R_f(\tau)\, R_f^+(t)\rangle_\Gamma &= \int_{-\infty}^{+\infty} J_{R_f^+,R_f}(\omega)\, e^{\omega/\theta}\, e^{i\omega(t-\tau)}\, d\omega. \end{aligned} \tag{1.44}$$

Here, R_f and R_f^+ are the same as in eqns. (1.20) and (1.22).

$$\begin{aligned} R_f^+(t) &= -i\,\frac{d\alpha_f^+(t)}{dt} - E(f)\, \alpha_f^+(t), \\ R_f(\tau) &= i\,\frac{d\alpha_f(\tau)}{d\tau} - E(f)\, \alpha_f(\tau). \end{aligned}$$

It is not difficult to see from this that

$$R_f^+(t)\,e^{-iE(f)\,t} = -i\,\frac{de^{-iE(f)\,t}\alpha_f^+(t)}{dt},$$

$$R_f(\tau)\,e^{iE(f)\,\tau} = i\,\frac{de^{iE(f)\,\tau}\alpha_f(\tau)}{d\tau}.$$

Therefore, on the basis of (1.44),

$$\int_{-\infty}^{+\infty} J_{R_f^+,\,R_f}(\omega)\,e^{i(\omega-E(f))(t-\tau)}\,d\omega = \left\langle\left(\frac{de^{-iE(f)\,t}\alpha_f^+(t)}{dt}\right)\left(\frac{de^{iE(f)\,\tau}\alpha_f(\tau)}{d\tau}\right)\right\rangle_\Gamma.$$

We integrate this equality over t and τ in the intervals $(t,\ t+\Delta)$ and $(\tau,\ \tau+\Delta)$ respectively, where Δ is an arbitrary number. We obtain

$$\int_{-\infty}^{+\infty} J_{R_f^+,\,R_f}(\omega)\,\frac{|e^{i(\omega-E(f))\,\Delta}-1|^2}{(\omega-E(f))^2}\,e^{i(\omega-E(f))(t-\tau)}\,d\omega$$
$$= \langle\{e^{-iE(f)(t+\Delta)}\alpha_f^+(t+\Delta)-e^{-iE(f)\,t}\alpha_f^+(t)\}$$
$$\times\{e^{iE(f)(\tau+\Delta)}\alpha_f(\tau+\Delta)-e^{iE(f)\,\tau}\alpha_f(\tau)\}\rangle_\Gamma.$$

But the right-hand side of this equality can also be determined from the formula (1.43). As a result, we obtain

$$\int_{-\infty}^{+\infty} J_{R_f^+,\,R_f}(\omega)\,\frac{|e^{i(\omega-E(f))\,\Delta}-1|^2}{(\omega-E(f))^2}\,e^{i(\omega-E(f))(t-\tau)}\,d\omega$$
$$= \int_{-\infty}^{+\infty} J_{\alpha_f^+,\,\alpha_f}(\omega)\,|e^{i(\omega-E(f))\,\Delta}-1|^2\,e^{i(\omega-E(f))(t-\tau)}\,d\omega.$$

Since this equality holds for arbitrary values of $(t-\tau)$, we see that

$$J_{R_f^+,\,R_f}(\omega) = J_{\alpha_f^+,\,\alpha_f}(\omega)\,(\omega-E(f))^2.$$

Therefore, from (1.42),

$$\int_{-\infty}^{+\infty} J_{\alpha_f^+,\alpha_f}(\omega)\,(\omega-E(f))^2\,d\omega = \langle R_f^+R_f\rangle_\Gamma,$$

$$\int_{-\infty}^{+\infty} J_{\alpha_f^+,\alpha_f}(\omega)\,e^{\omega/\theta}(\omega-E(f))^2\,d\omega = \langle R_fR_f^+\rangle_\Gamma.$$

Taking notice of the inequalities (1.30), we see from this that

$$\begin{aligned}
&\int_{-\infty}^{+\infty} J_{\alpha_f^+,\alpha_f}(\omega)\,(\omega-E(f))^2\,d\omega \leq 2M_1\left\{\varepsilon_V+\frac{M_2}{4V}\right\},\\
&\int_{-\infty}^{+\infty} J_{\alpha_f^+,\alpha_f}(\omega)\,e^{\omega/\theta}(\omega-E(f))^2\,d\omega \leq 2M_1\left\{\varepsilon_V+\frac{M_2}{4V}\right\}.
\end{aligned} \tag{1.45}$$

Following the basic idea of the method described in the paper by Bogolyubov, Jr. (1968e), we now evaluate the difference

$$\mathcal{D} = \langle\alpha_f^+B\rangle_\Gamma - e^{-E(f)/\theta}\langle B\alpha_f^+\rangle_\Gamma, \tag{1.46}$$

where B is a product of an even number of operators β_{f_j}, arranged in "normal" order, i.e. with all the α_f^+ to the left and all the α_f to the right,

$$B = \ldots\, \alpha_{f_j}^+ \,\ldots\, \alpha_{f_k'} \,\ldots,$$

and establish the asymptotic smallness of this difference.

We note, first, that we shall need to consider only the situation when no two f_j are the same and no two f_k' are the same, since otherwise B is found to be identically zero. But in such a situation, if we take into account that the total number of operators β_f in B is odd, at least one of the following statements is true:

1. In the operator product B there is one operator $\alpha_{f_i}^+$ such that the index f_i is not equal to any of the indices f of the other operators β_f contained in B.

2. In the operator product B there is one operator α_{f_k} such that the index f_k is not equal to any of the indices f of the other operators β_f contained in B.

For example, let statement (1) be true. Then we can write that

$$B = \alpha_{f_i}^{+} W,$$

where W is the product of the remaining β_f in B.

$$W = \ldots \beta_f \ldots \tag{1.47}$$

Since the number of operators occurring in W is even and their indices f all differ from f_i, we see that $\alpha_{f_i}^{+}$ must commute with W:

$$B = \alpha_{f_i}^{+} W = W\alpha_{f_i}^{+}.$$

Taking the conjugate, we also find

$$B^{+} = W^{+}\alpha_{f_i} = \alpha_{f_i}W^{+}.$$

On the other hand, it follows immediately from (1.47) that

$$W^{+}W \leqslant 1, \quad WW^{+} \leqslant 1.$$

Therefore,

$$B^{+}B = \alpha_{f_i}W^{+}W\alpha_{f_i}^{+} \leqslant \alpha_{f_i}\alpha_{f_i}^{+},$$

$$BB^{+} = \alpha_{f_i}^{+}WW^{+}\alpha_{f_i} \leqslant \alpha_{f_i}^{+}\alpha_{f_i}.$$

Hence,

$$B^{+}B+BB^{+} \leqslant 1.$$

We can convince ourselves of the validity of this operator inequality in case (2) in a completely analogous way. Thus,

$$\langle B^{+}B+BB^{+}\rangle_{\Gamma} \leqslant 1$$

always. Hence it follows that

$$\int_{-\infty}^{+\infty} J_{B^{+},B}(\omega)(1+e^{\omega/\theta})\,d\omega \leqslant 1. \tag{1.48}$$

We turn now to the quantity (1.46). Using the above spectral representations, we write

$$\mathcal{D} = \int_{-\infty}^{+\infty} J_{\alpha_f^+, B}(\omega) \{1 - e^{\{\omega - E(f)\}/\theta}\} \, d\omega \tag{1.49}$$

and

$$|\mathcal{D}| \leqslant \int_{-\infty}^{+\infty} \left| J_{\alpha_f^+, B}(\omega) \right| \, |1 - e^{\{\omega - E(f)\}/\theta}| \, d\omega. \tag{1.50}$$

We note here that if

$$\omega \geqslant E(f),$$

then

$$|1 - e^{\{\omega - E(f)\}/\theta}| = e^{\{\omega - E(f)\}/\theta} - 1 \leqslant \frac{\omega - E(f)}{\theta} e^{\{\omega - E(f)\}/\theta}$$

$$\leqslant \frac{(\omega - E(f))}{\theta} e^{\omega/\theta},$$

since, by definition,

$$E(f) \geqslant 0 \tag{1.51}$$

always. Further, if

$$\omega < E(f),$$

then

$$|1 - e^{\{\omega - E(f)\}/\theta}| = 1 - e^{-\{E(f) - \omega\}/\theta} \leqslant \frac{E(f) - \omega}{\theta}.$$

Thus,

$$|1 - e^{\{\omega - E(f)\}/\theta}| \leqslant \frac{|\omega - E(f)|}{\theta} (1 + e^{\omega/\theta})$$

always, in view of which we obtain from (1.50)

$$|\mathcal{D}| \leqslant \frac{1}{\theta} \int_{-\infty}^{+\infty} \left| J_{\alpha_f^+, B}(\omega) \right| \, |\omega - E(f)| \, (1 + e^{\omega/\theta}) \, d\omega.$$

Hence, using the inequality (1.42), we obtain

$$|\mathcal{D}| \leqslant \frac{1}{\theta} \sqrt{\int\limits_{-\infty}^{+\infty} J_{\alpha_f^+, \alpha_f}(\omega)\,(\omega - E(f))^2\,(1 + e^{\omega/\theta})\,d\omega}$$

$$\times \sqrt{\int\limits_{-\infty}^{+\infty} J_{B^+, B}(\omega)\,(1 + e^{\omega/\theta})\,d\omega}.$$

Bearing in mind the inequalities (1.45) and (1.48), we therefore obtain

$$|\mathcal{D}| \leqslant \frac{2}{\theta} \sqrt{M_1\left(\varepsilon_V + \frac{M_2}{4V}\right)}. \tag{1.52}$$

§ 5. REMARK (I)

This bound was obtained by starting from the trivial property (1.51). Since θ occurs in the denominator of (1.52), we cannot use this bound for treating the zero-temperature limit $\theta \to 0$, even if $\varepsilon_V \to 0$ $(V \to \infty)$ uniformly with $\theta \to 0$.

Let

$$E(f) \geqslant \gamma, \tag{1.53}$$

for the (f) under consideration, where γ is a certain positive constant. Since

$$E(f) = \sqrt{\left(\frac{p^2}{2m} - \mu\right)^2 + |\Delta(f)|^2},$$

we see that the inequality (1.53) is always satisfied outside the spherical layer:

$$\left|\frac{p^2}{2m} - \mu\right| \leqslant \Delta.$$

We turn now to the inequality (1.50). We note from what has been said above that

$$|1 - e^{\{\omega - E(f)\}/\theta}| \leqslant \frac{|\omega - E(f)|}{\theta}\, e^{\omega/\theta} e^{((2/3)\gamma - E(f))/\theta}$$

for

$$\omega \geqslant E(f)-\tfrac{2}{3}\gamma.$$

Also,

$$|1-e^{\{\omega-E(f)\}/\theta}| \leqslant 1$$

for the values

$$\omega \leqslant E(f)-\tfrac{2}{3}\gamma.$$

Thus, from (1.50) we find

$$|\mathcal{D}| \leqslant \int\limits_{E-2\gamma/3}^{\infty} \left|J_{\alpha_f^+, B}(\omega)\right| |\omega-E(f)|\, e^{\omega/\theta}\, d\omega\, \frac{1}{\theta}\, e^{-\gamma/3\theta} + \int\limits_{-\infty}^{E-2\gamma/3} \left|J_{\alpha_f^+, B}(\omega)\right| d\omega.$$

$$|\mathcal{D}| \leqslant \frac{1}{\theta}\, e^{-\gamma/3\theta} \sqrt{\int\limits_{-\infty}^{+\infty} J_{\alpha_f^+, \alpha_f}(\omega)\, \{\omega-E(f)\}^2 e^{\omega/\theta}\, d\omega}$$

$$\times \sqrt{\int\limits_{-\infty}^{+\infty} J_{B^+, B}(\omega) e^{\omega/\theta}\, d\omega} + \sqrt{\int\limits_{-\infty}^{E-2\gamma/3} J_{\alpha_f^+, \alpha_f}(\omega)\, d\omega \int\limits_{-\infty}^{+\infty} J_{B^+, B}(\omega)\, d\omega}.$$

But it follows from (1.45) that

$$\int\limits_{-\infty}^{E-2\gamma/3} J_{\alpha_f^+, \alpha_f}(\omega)\, d\omega \leqslant \frac{1}{(2\gamma/3)^2}\, 2M_1\left(\varepsilon_V+\frac{M_2}{4V}\right).$$

Therefore, we obtain

$$|\mathcal{D}| \leqslant \left\{\frac{1}{\theta}\, e^{-\gamma/3\theta} \sqrt{\langle BB^+\rangle_\Gamma}+\frac{3}{2\gamma} \sqrt{\langle B^+B\rangle_\Gamma}\right\} \sqrt{2M_1\left(\varepsilon_V+\frac{M_2}{4V}\right)}.$$

Hence, taking into account that

$$\langle BB^+ + B^+B\rangle_\Gamma \leqslant 1,$$

we find

$$|\mathcal{D}| \leqslant \sqrt{\frac{1}{\theta^2}e^{-2\gamma/3\theta}+\left(\frac{3}{2\gamma}\right)^2}\sqrt{2M_1\left(\varepsilon_V+\frac{M_2}{4V}\right)}.$$

But it can be seen that

$$x^2e^{-x} \leqslant \left(\frac{2}{e}\right)^2$$

and therefore

$$\frac{1}{\theta^2}e^{-2\gamma/3\theta} = \left(\frac{2\gamma}{3\theta}\right)^2 e^{-2\gamma/3\theta}\left(\frac{3}{2\gamma}\right)^2 \leqslant \left(\frac{2}{e}\right)^2\left(\frac{3}{2}\gamma\right)^2.$$

Thus, finally,

$$|\mathcal{D}| \leqslant \frac{3}{2\gamma}\sqrt{2+2\left(\frac{2}{e}\right)^2}\sqrt{M_1\left(\varepsilon_V+\frac{M_2}{4V}\right)}. \tag{1.54}$$

we combine the inequalities (1.52) and (1.54) in the form

$$|\mathcal{D}| \leqslant \mathcal{K}\sqrt{\varepsilon_V+\frac{M_2}{4V}}, \tag{1.55}$$

where the constant $\mathcal{K}$ can always be taken in the form

$$\mathcal{K} = \frac{2}{\theta}\sqrt{M_1}. \tag{1.56}$$

In the region of (f) for which the inequality (1.53) is fulfilled, we can also put

$$\mathcal{K} = \frac{3}{2\gamma}\left\{2M_1+2\left(\frac{2}{e}\right)^2 M_1\right\}^{1/2}. \tag{1.57}$$

This latter value of $\mathcal{K}$ clearly no longer depends in any way on the temperature θ. Thus, we can write

$$|\langle\alpha_f^+B\rangle_\Gamma - e^{-E(f)/\theta}\langle B\alpha_f^+\rangle_\Gamma| \leqslant \mathcal{K}\sqrt{\varepsilon_V+\frac{M_2}{4V}}.$$

Hence follows the inequality

$$|(1+e^{-E(f)/\theta})\langle\alpha_f^+B\rangle_\Gamma - e^{-E(f)/\theta}\langle\alpha_f^+B+B\alpha_f^+\rangle_\Gamma| \leqslant \mathcal{K}\sqrt{\varepsilon_V+\frac{M_2}{4V}},$$

and therefore

$$\left|\langle\alpha_f^+ B\rangle_\Gamma - \frac{e^{-E(f)/\theta}}{1+e^{-E(f)/\theta}}\langle\alpha_f^+ B + B\alpha_f^+\rangle_\Gamma\right| \leqslant \mathcal{K}\sqrt{\varepsilon_V + \frac{M_2}{4V}}\,. \tag{1.58}$$

From this inequality we shall now be able to obtain the required estimates for the difference (1.39).

§ 6. PROOF OF THE CLOSENESS OF AVERAGES CONSTRUCTED ON THE BASIS OF MODEL AND TRIAL HAMILTONIANS FOR "NORMAL" ORDERING OF THE OPERATORS IN THE AVERAGES

We shall first of all examine an operator product of the form

$$U = \alpha_{f_1}^+ \ldots \alpha_{f_k}^+ \alpha_{f_1'} \ldots \alpha_{f_q'},$$

in which

$$k > q \geqslant 0; \qquad k+q \text{ is an even number.}$$

We assume, of course, that all the f_i (like all the f_i') are different, since otherwise U will be identically zero. Now, as soon as the number of creation operators α^+ in a given product is greater than the number of annihilation operators α, there is certain to be amongst them an operator $\alpha_{f_j}^+$ such that the index f_j is not equal to any of the indices $f_1', \ldots, f_q'$. But then we can write

$$U = \alpha_{f_j}^+ B,$$

where B is the operator product of the remaining β_f. The essential point is that the number of operators β_f in the product B is odd and that their indices all differ from the index f_j.

Therefore, the operator $\alpha_{f_j}^+$ must anticommute with the product B:

$$\alpha_{f_j}^+ B + B\alpha_{f_j}^+ = 0.$$

Using the inequality (1.58), we obtain the inequality

$$\left|\langle \overset{+}{\alpha}_{f_1} \ldots \overset{+}{\alpha}_{f_k} \alpha_{f'_1} \ldots \alpha_{f'_q} \rangle_\Gamma\right| \leqslant \mathcal{K}\sqrt{\varepsilon_V + \frac{M_2}{4V}}, \tag{1.59}$$

$$k > q \geqslant 0; \qquad k+q \ \text{even.}$$

We now take the opposite case. Let

$$U_1 = \overset{+}{\alpha}_{f_1} \ldots \overset{+}{\alpha}_{f_k} \alpha_{f'_1} \ldots \alpha_{f'_q},$$

$$q > k \geqslant 0; \qquad k+q \ \text{even.}$$

We have

$$U_1^+ = \overset{+}{\alpha}_{f'_q} \ldots \overset{+}{\alpha}_{f'_1} \alpha_{f_k} \ldots \alpha_{f_1}$$

and for U^+ we can again make use of the inequality (1.59). But

$$|\langle U_1 \rangle_\Gamma| = |\langle U \rangle_\Gamma^*| = |\langle U^+ \rangle_\Gamma|,$$

and therefore

$$\left|\langle \overset{+}{\alpha}_{f_1} \ldots \overset{+}{\alpha}_{f_k} \alpha_{f'_1} \ldots \alpha_{f'_q} \rangle_\Gamma\right| \leqslant \mathcal{K}\sqrt{\varepsilon_V + \frac{M_2}{4V}}, \tag{1.60}$$

$$q > k \geqslant 0; \qquad k+q \ \text{even.}$$

We now consider the case when $k = q$:

$$U = \overset{+}{\alpha}_{f_1} \ldots \overset{+}{\alpha}_{f_k} \alpha_{f'_1} \ldots \alpha_{f'_k}.$$

If amongst the indices $f_1, \ldots, f_k$ in this product there is at least one index f_i which is not equal to any of the indices $f'_1, \ldots, f'_k$ we have, analogously,

$$U = \overset{+}{\alpha}_{f_i} B; \qquad \overset{+}{\alpha}_{f_i} B + B \overset{+}{\alpha}_{f_i} = 0,$$

and from the inequality (1.58) it follows again that

$$\left|\langle \overset{+}{\alpha}_{f_1} \ldots \overset{+}{\alpha}_{f_k} \alpha_{f'_1} \ldots \alpha_{f'_k} \rangle_\Gamma\right| \leqslant \mathcal{K}\sqrt{\varepsilon_V + \frac{M_2}{4V}} \tag{1.61}$$

(if any one of the indices $f_1, \ldots, f_k$ is not equal to any of the indices $f'_1, \ldots, f'_k$).

Since in the "new" Fermi amplitudes the trial Hamiltonian Γ_a has the form

$$\Gamma_a = \sum_f E(f)\,\alpha^+_f \alpha_f + \text{const},$$

we see that the averages examined above, constructed on the basis of the Hamiltonian Γ in the cases (1.59), (1.60) and (1.61), will be equal to zero if they are calculated on the basis of the trial Hamiltonian Γ_a.

Consequently, the inequalities (1.59), (1.60) and (1.61) can be rewritten in the form:

$$|\langle\ldots\rangle_\Gamma - \langle\ldots\rangle_{\Gamma_a}| \leq \mathcal{K}\sqrt{\varepsilon_V + \frac{M_2}{4V}}\,, \tag{1.62}$$

where the averages are of the products treated earlier.

Thus, of all the products

$$\beta_{f_1} \ldots \beta_{f_s} \qquad (s \text{ even})$$

in which the β_f are arranged in *normal* order (i.e. with all creation operators α^+_f to the left and all annihilation operators α_f to the right) we shall only have to consider products

$$\alpha^+_{f_1} \ldots \alpha^+_{f_k} \alpha_{f'_1} \ldots \alpha_{f'_k}$$

in which each of the indices $f_1, \ldots, f_k$ is equal to one of the indices $f'_1, \ldots, f'_k$. But such products can obviously be reduced to the form

$$\pm \alpha^+_{f_k} \ldots \alpha^+_{f_1} \alpha_{f_1} \ldots \alpha_{f_k}.$$

We put

$$\mathcal{U}_k = \alpha^+_{f_k} \ldots \alpha^+_{f_1} \alpha_{f_1} \ldots \alpha_{f_k} = \alpha^+_{f_k} B, \tag{1.63}$$

and note (all the indices $f_1, \ldots, f_k$ are different!) that

$$\begin{aligned} \alpha^+_{f_k} B_k + B_k \alpha^+_{f_k} &= \alpha^+_{f_k} \alpha_{f_k} \alpha^+_{f_{k-1}} \ldots \alpha^+_{f_1} \alpha_{f_1} \ldots \alpha_{f_{k-1}} \\ &\quad + \alpha_{f_k} \alpha^+_{f_k} \alpha^+_{f_{k-1}} \ldots \alpha^+_{f_1} \alpha_{f_1} \ldots \alpha_{f_{k-1}} = \mathcal{U}_{k-1} \end{aligned}$$

Therefore, from the inequality (1.58) we shall have

$$\left|\langle\mathcal{U}_k\rangle_\Gamma-\frac{1}{1+e^{E(f_k)/\theta}}\langle\mathcal{U}_{k-1}\rangle_\Gamma\right|\leqslant\mathcal{K}\sqrt{\varepsilon_V+\frac{M_2}{4V}}, \tag{1.64}$$

and, in particular, for

$$\mathcal{U}_1=\overset{+}{\alpha}_{f_1}\alpha_{f_1},$$

we find

$$\left|\left\langle\overset{+}{\alpha}_{f_1}\alpha_{f_1}\right\rangle_\Gamma-\frac{1}{1+e^{\{E(f_1)\}/\theta}}\right|\leqslant\mathcal{K}\sqrt{\varepsilon_V+\frac{M_2}{4V}}. \tag{1.65}$$

But, obviously,

$$\frac{1}{1+e^{\{E(f_1)\}/\theta}}=\left\langle\overset{+}{\alpha}_{f_1}\alpha_{f_1}\right\rangle_{\Gamma_0}.$$

Also,

$$\langle\mathcal{U}_k\rangle_{\Gamma_0}-\frac{1}{1+e^{\{E(f_k)\}/\theta}}\langle\mathcal{U}_{k-1}\rangle_{\Gamma_0}=0. \tag{1.66}$$

Therefore, it follows from (1.64) that

$$\left|\langle\mathcal{U}_k\rangle_\Gamma-\langle\mathcal{U}_k\rangle_{\Gamma_0}-\frac{1}{1+e^{\{E(f_k)\}/\theta}}\left\{\langle\mathcal{U}_{k-1}\rangle_\Gamma-\langle\mathcal{U}_{k-1}\rangle_{\Gamma_0}\right\}\right|$$
$$\leqslant\mathcal{K}\sqrt{\varepsilon_V+\frac{M_2}{4V}}.$$

Hence, transforming, we have

$$\begin{aligned}|\langle\mathcal{U}_k\rangle_\Gamma-\langle\mathcal{U}_k\rangle_{\Gamma_0}|\leqslant\frac{1}{2}&|\langle\mathcal{U}_{k-1}\rangle_\Gamma-\langle\mathcal{U}_{k-1}\rangle_{\Gamma_0}|\\&+\mathcal{K}\sqrt{\varepsilon_V+\frac{M_2}{4V}}.\end{aligned} \tag{1.67}$$

It follows from the inequality (1.65) that

$$|\langle\mathcal{U}_1\rangle_\Gamma-\langle\mathcal{U}_1\rangle_{\Gamma_0}|\leqslant\mathcal{K}\sqrt{\varepsilon_V+\frac{M_2}{4V}}, \tag{1.68}$$

and from the inequalities (1.67) and (1.68) that

$$|\langle \mathcal{U}_k\rangle_\Gamma - \langle \mathcal{U}_k\rangle_{\Gamma_a}| \leq \mathcal{K}\sqrt{\varepsilon_V + \frac{M_2}{4V}}\left\{1+\frac{1}{2}+\ \dots\ +\left(\frac{1}{2}\right)^{k-1}\right\}$$
$$< 2\mathcal{K}\sqrt{\varepsilon_V + \frac{M_2}{4V}}\,.$$

Thus, to summarize the above statements, we have shown that for products

$$\beta_{f_1}\dots\beta_{f_s} \tag{1.69}$$

in which the operators β_f are normally ordered, the inequalities

$$|\langle \beta_{f_1}\dots\beta_{f_s}\rangle_\Gamma - \langle \beta_{f_1}\dots\beta_{f_s}\rangle_{\Gamma_a}| < 2\mathcal{K}\sqrt{\varepsilon_V + \frac{M_2}{4V}} \tag{1.70}$$

are valid.

§ 7. PROOF OF THE CLOSENESS OF THE AVERAGES FOR ARBITRARY ORDERING OF THE OPERATORS IN THE AVERAGES

We shall now consider the case of products (1.69) in which the operators β_f are arranged in arbitrary order. We note that in the system of indices $f_1, \dots, f_s$, some of them can be equal.

From the aggregate of indices $f_1, \dots, f_s$, we separate out all the distinct indices and denote them by $f'_1, \dots, f'_k$, so that each (one or several) of the indices $f_1, \dots, f_s$ is equal to one of the indices $f'_1, \dots, f'_k$.

We shall represent the operator product under consideration in the form

$$\beta_{f'_1}\dots\beta_{f'_k}, \tag{1.71}$$

where $B_{f'}$ denotes a product of operators $\alpha_{f'}$, $\alpha^+_{f'}$, all with the same index f'.

However, by virtue of the commutation properties of the Fermi operators, it is not difficult to see that $B_{f'}$ can always be reduced to one of the following forms:

$$B_{f'} = \pm\alpha_{f'}, \quad B_{f'} = \pm\alpha_{f'}^{+}, \quad B_{f'} = \pm\alpha_{f'}^{+}\alpha_{f'}, \tag{1.72}$$

$$B_{f'} = \pm\alpha_{f'}\alpha_{f'}^{+} = \pm(1-\alpha_{f'}^{+}\alpha_{f'}), \tag{1.73}$$

leaving aside the trivial case when $B_{f'}$ is identically zero.

We shall reduce the products (1.71) to normal order, commuting the operators $\alpha_{f'}^{+}$ only to the left and the operators $\alpha_{f'}$ only to the right. The number of terms will not "multiply" as a result of the commutation operation, since all the indices $f_1', \ldots, f_k'$ are different, and $B_{f'}$ consists of a sum of two terms in the cases (1.73) only.

Therefore, the number of products of normally ordered operators into which the original product breaks down will be 2^q, where q is the number of $B_{f'}$ having the form (1.73).

But, obviously, $q \leqslant s/2$. Therefore, by making use of the inequality (1.70) for each of the "normal products", we shall obtain for the general case under consideration:

$$|\langle\beta_{f_1} \ldots \beta_{f_s}\rangle_{\Gamma} - \langle\beta_{f_1} \ldots \beta_{f_s}\rangle_{\Gamma_a}| \leqslant 2^{(s/2)+1}\mathcal{K}\sqrt{\varepsilon_V + \frac{M_2}{4V}}\,. \tag{1.74}$$

To summarize the arguments given above, we shall formulate a theorem:

THEOREM I. *Let the conditions* I *(Chapter* 1, § 1*) be fulfilled. Then the bound* (1.74) *will be valid for the difference of singletime averages constructed on the basis of the Hamiltonians* Γ *and* Γ_a.

It must be emphasized that in this bound we can always take the value (1.56) for $\mathcal{K}$:

$$\mathcal{K} = \frac{2}{\theta}\sqrt{M_1}.$$

But then the estimate (1.74) does not allow us, in general, to pass to the treatment of the case $\theta \to 0$, even if

$$\varepsilon_V \to 0 \quad \text{as} \quad V \to \infty$$

uniformly with respect to the temperature θ in the interval $0 < \theta \leqslant \theta_0$, where θ_0 is any fixed temperature.

On the other hand, from what we said earlier we can use a "uniform estimate" for $\mathcal{K}$ by taking the value (1.57) of the constant,

$$\mathcal{K} = \frac{3}{2\gamma}\left\{2M_1+2\left(\frac{2}{e}\right)^2 M_1\right\}^{1/2},$$

if all the values of $f_1, f_2, \ldots, f_s$ lie in the region in which

$$E(f) \geqslant \gamma.$$

REMARK (II)

We note, in passing, that it would be incorrect to think that the difference (1.74) in fact always tends to zero (as $V \to \infty$) uniformly with respect to θ ($\theta > 0$), even in the region where $E(f) = 0$; the absence of this uniformity is due entirely to our method of treatment, which permits excessive majorization.

To understand intuitively the reason for the absence of uniformity with respect to θ in the region where $E(f) = 0$, we take an extremely simple (one could even say trivial) example:

$$\Gamma = \Gamma_a + \Gamma_1$$

where

$$\Gamma_a = \sum_f E(f)\alpha_f^\dagger\alpha_f; \quad E(f) = \left|\frac{p^2}{2m}-\mu\right|,$$

$$\Gamma_1 = \sum_f \varepsilon\alpha_f^\dagger\alpha_f; \qquad \varepsilon > 0.$$

Here,

$$\varepsilon \to 0 \quad \text{as} \quad V \to \infty.$$

It is clear here that binary averages calculated over the Hamiltonians Γ and Γ_a have the form

$$\langle\alpha_f^\dagger\alpha_f\rangle_\Gamma = \frac{1}{1+e^{(E(f)+\varepsilon)/\theta}},$$

$$\langle\alpha_f^\dagger\alpha_f\rangle_{\Gamma_a} = \frac{1}{1+e^{E(f)/\theta}}.$$

We take some value $f = f_0$ such that

$$E(f_0) = \frac{p_0^2}{2m} - \mu = 0.$$

Then

$$\langle \alpha_{f_0}^{+} \alpha_{f_0} \rangle_{\Gamma} - \langle \alpha_{f_0}^{+} \alpha_{f_0} \rangle_{\Gamma_a} = \frac{1}{1+e^{\varepsilon/\theta}} - \frac{1}{2}.$$

Putting here

$$\theta = 2\varepsilon > 0,$$

we find

$$\langle \alpha_{f_0}^{+} \alpha_{f_0} \rangle_{\Gamma} - \langle \alpha_{f_0}^{+} \alpha_{f_0} \rangle_{\Gamma_a} = \frac{1}{1+\sqrt{e}} - \frac{1}{2}.$$

Thus, even in this simplest case, we see that the asymptotic relation

$$\langle \alpha_f^{+} \alpha_f \rangle_{\Gamma} - \langle \alpha_f^{+} \alpha_f \rangle_{\Gamma_a} \to 0$$

is not uniform with respect to $\theta \to 0$ in the region $E(f) = 0$.

§ 8. ESTIMATES OF THE ASYMPTOTIC CLOSENESS OF THE MANY-TIME CORRELATION AVERAGES

We turn now to consider "many-time averages"; we shall estimate differences of the form

$$\langle \beta_{f_1}(t_1) \ldots \beta_{f_s}(t_s) \rangle_{\Gamma} - \langle \beta_{f_1}(t_1) \ldots \beta_{f_s}(t_s) \rangle_{\Gamma_a}.$$

Here we use the following conventions:

1. Operators $\beta_f(t)$ occurring in the average

$$\langle \ldots \rangle_{\Gamma}$$

are determined by the equations of motion for the Hamiltonian Γ with the initial conditions

$$\beta_f(0) = \beta_f.$$

2. Operators $\beta_f(t)$ occurring in the average

$$\langle \ldots \rangle_{\Gamma_a}$$

are determined by the equations of motion for the Hamiltonian Γ_a with the same initial conditions.

We note that operators $\beta_f(t)$ for the trial Hamiltonian Γ_a can be represented in a form with the time-dependence given explicitly:

$$\beta_f(t) = \beta_f e^{-i\omega(f)t},$$

where

$$\omega(f) = E(f) \quad \text{if} \quad \beta_f = \alpha_f,$$
$$\omega(f) = -E(f) \quad \text{if} \quad \beta_f = \alpha_f^+.$$

Assuming that the conditions I are fulfilled, we establish the validity of the inequality[†]

$$|\langle\beta_{f_1}(t_1)\ \dots\ \beta_{f_s}(t_s)\rangle_\Gamma - \langle\beta_{f_1}(t_1)\ \dots\ \beta_{f_s}(t_s)\rangle_{\Gamma_a}|$$
$$\leq \eta_V^{(s)} \sum_{j=1}^{s-1} j|t_j - t_{j+1}| + 2^{(s/2)+1}\mathcal{K}\sqrt{\varepsilon_V + \frac{M_2}{4V}}, \tag{1.75}$$

where

$$\eta_V^{(j)} = \sqrt{2M_1}\left\{\frac{M_2}{4V} + \left(\sqrt{\varepsilon_V} + \frac{(j-1)}{V}\sqrt{M_1}\right)^2\right\}^{1/2} \tag{1.76}$$
$$(j = 1, 2, 3, \dots).$$

We note, first of all, that since neither of the Hamiltonians Γ and Γ_a depends explicitly on time, both the averages considered will be invariant with respect to time translations

$$t \to t + \tau$$

with arbitrary τ. Taking $\tau = -t_s$, we obtain a situation in which t_s is replaced by 0.

Therefore, the inequality (1.75) will be proved to be true if we prove it only for the particular case $t_s = 0$. Taking into account the nature of the time-dependence of the operators $\beta_f(t)$ for the trial Hamiltonian

---

[†] On transforming to the old Fermi operators, we prove an analogous inequality by formulating a theorem (cf. Theorem II, p. 64).

Γ_a, we see that we must prove the inequality

$$\begin{aligned}
&|\langle\beta_{f_1}(t_1)\,\ldots\,\beta_{f_{s-1}}(t_{s-1})\,\beta_{f_s}(0)\rangle_\Gamma \\
&\quad -e^{-i\{\omega(f_1)t_1+\,\ldots\,+\omega(f_{s-1})t_{s-1}\}}\langle\beta_{f_1}\,\ldots\,\beta_{f_s}\rangle_{\Gamma_a}| \\
&\leqslant \eta_V^{(s)}\{|t_1-t_2|+\,\ldots\,+(s-2)\,|t_{s-2}-t_{s-1}|+(s-1)\,|t_{s-1}|\} \\
&\quad +2^{(s/2)+1}\mathcal{K}\sqrt{\varepsilon_V+\frac{M_2}{4V}}\,.
\end{aligned}\tag{1.77}$$

But

$$\begin{aligned}
&|\langle\beta_{f_1}(t_1)\,\ldots\,\beta_{f_{s-1}}(t_{s-1})\,\beta_{f_s}(0)\rangle_\Gamma \\
&\quad -e^{-i\{\omega(f_1)t_1+\,\ldots\,+\omega(f_{s-1})t_{s-1}\}}\langle\beta_{f_1}\,\ldots\,\beta_{f_s}\rangle_{\Gamma_a}| \\
&= |\langle\beta_{f_1}(t_1)\,\ldots\,\beta_{f_{s-1}}(t_{s-1})\,\beta_{f_s}(0)\rangle_\Gamma e^{i\{\omega(f_1)t_1+\,\ldots\,+\omega(f_{s-1})t_{s-1}\}} \\
&\quad -\langle\beta_{f_1}\,\ldots\,\beta_{f_s}\rangle_{\Gamma_a}| \\
&= \left|\langle\tilde{\beta}_{f_1}(t_1)\,\ldots\,\tilde{\beta}_{f_{s-1}}(t_{s-1})\,\beta_{f_s}(0)\rangle_\Gamma-\langle\beta_{f_1}\,\ldots\,\beta_{f_s}\rangle_{\Gamma_a}\right|,
\end{aligned}$$

where

$$\tilde{\beta}_f(t)=\beta_f(t)e^{i\omega(f)t}, \tag{1.78}$$

and, therefore, the inequality (1.77) is equivalent to the inequality

$$\begin{aligned}
&|\langle\tilde{\beta}_{f_1}(t_1)\,\ldots\,\tilde{\beta}_{f_{s-1}}(t_{s-1})\,\beta_{f_s}(0)\rangle_\Gamma-\langle\beta_{f_1}\,\ldots\,\beta_{f_s}\rangle_{\Gamma_a}| \\
&\quad\leqslant \eta_V^{(s)}\{|t_1-t_2|+\,\ldots\,+(s-2)\,|t_{s-2}-t_{s-1}|+(s-1)\,|t_{s-1}|\} \\
&\qquad +2^{(s/2)+1}\mathcal{K}\sqrt{\varepsilon_V+\frac{M_2}{4V}}\,.
\end{aligned}\tag{1.79}$$

It is not difficult to see that this inequality (1.79) will be proved as soon as we have established that

$$\begin{aligned}
&|\langle\tilde{\beta}_{f_1}(t_1)\,\ldots\,\tilde{\beta}_{f_{s-1}}(t_{s-1})\beta_{f_s}(0)\rangle_\Gamma-\langle\beta_{f_1}(0)\,\ldots\,\beta_{f_{s-1}}(0)\beta_{f_s}(0)\rangle_\Gamma| \\
&\quad\leqslant \eta_V^{(s)}\{|t_1-t_2|+\,\ldots\,+(s-2)\,|t_{s-2}-t_{s-1}|+(s-1)\,|t_{s-1}|\}.
\end{aligned}\tag{1.80}$$

In fact, if the inequality (1.80) is true, the left-hand side of inequality (1.79) will be less than the left-hand side of inequality (1.80) plus the expression

$$\begin{aligned}
&|\langle\beta_{f_1}(0)\,\ldots\,\beta_{f_s}(0)\rangle_\Gamma-\langle\beta_{f_1}\,\ldots\,\beta_{f_s}\rangle_{\Gamma_a}| \\
&\quad= |\langle\beta_{f_1}\,\ldots\,\beta_{f_s}\rangle_\Gamma-\langle\beta_{f_1}\,\ldots\,\beta_{f_s}\rangle_{\Gamma_a}|,
\end{aligned}$$

which, by virtue of the inequality (1.74), will be

$$\leqslant 2^{(s/2)+1}\mathcal{K}\sqrt{\varepsilon_V+\frac{M_2}{4V}}\,.$$

To establish the inequality (1.80), we note that

$$\begin{aligned}
&\left|\langle\tilde{\beta}_{f_1}(t_1)\ \dots\ \tilde{\beta}_{f_{s-1}}(t_{s-1})\,\beta_{f_s}(0)\rangle_\Gamma-\langle\beta_{f_1}(0)\ \dots\ \beta_{f_{s-1}}(0)\,\beta_{f_s}(0)\rangle_\Gamma\right|\\
&\quad=\Bigg|\sum_{j=1}^{s-1}\{\langle\tilde{\beta}_{f_1}(t_j)\ \dots\ \tilde{\beta}_{f_j}(t_j)\tilde{\beta}_{f_{j+1}}(t_{j+1})\ \dots\ \beta_{f_s}(0)\rangle_\Gamma\\
&\quad-\langle\tilde{\beta}_{f_1}(t_{j+1})\ \dots\ \tilde{\beta}_{f_j}(t_{j+1})\,\tilde{\beta}_{f_{j+1}}(t_{j+1})\ \dots\ \beta_{f_s}(0)\rangle_\Gamma\}\Bigg|\\
&\quad\leqslant\sum_{j=1}^{s-1}\left|\int\limits_{t_j}^{t_{j+1}}dt\left\langle\frac{d\tilde{\beta}_{f_1}(t)\ \dots\ \tilde{\beta}_{f_j}(t)}{dt}\ \mathcal{A}_j\right\rangle_\Gamma\right|,
\end{aligned}\tag{1.81}$$

where

$$\mathcal{A}_j=\tilde{\beta}_{f_{j+1}}(t_{j+1})\ \dots\ \beta_{f_s}(0).$$

In view of this,

$$\mathcal{A}_j^{+}\mathcal{A}_j\leqslant 1.\tag{1.82}$$

But,

$$\begin{aligned}
&\left|\int\limits_{t_j}^{t_{j+1}}dt\left\langle\frac{d\tilde{\beta}_{f_1}(t)\ \dots\ \tilde{\beta}_{f_j}(t)}{dt}\,\mathcal{A}_j\right\rangle_\Gamma\right|\\
&\qquad\leqslant\sum_{q=1}^{j}\left|\int\limits_{t_j}^{t_{j+1}}dt\left\langle\tilde{\beta}_{f_1}(t)\ \dots\ \frac{d\tilde{\beta}_{f_q}(t)}{dt}\,\mathcal{A}_{q,j}\right\rangle_\Gamma\right|,
\end{aligned}$$

where

$$\mathcal{A}_{q,j}=\tilde{\beta}_{f_{q+1}}(t)\ \dots\ \tilde{\beta}_{f_j}(t)\mathcal{A}_j,$$

and, from (1.82), it is clear that

$$\langle\mathcal{A}_{q,j}^{+}\mathcal{A}_{q,j}\rangle\leqslant 1.$$

Therefore,

$$\begin{aligned}
&\left|\left\langle\tilde{\beta}_{f_1}(t)\ \dots\ \frac{d\tilde{\beta}_{f_q}(t)}{dt}\,\mathcal{A}_{q,j}\right\rangle_\Gamma\right|\\
&\quad\leqslant\sqrt{\left\langle\tilde{\beta}_{f_1}(t)\ \dots\ \frac{d\tilde{\beta}_{f_q}(t)}{dt}\,\frac{d\tilde{\beta}^{+}_{f_q}(t)}{dt}\ \dots\ \tilde{\beta}_{f_1}(t)\right\rangle_\Gamma}
\end{aligned}$$

Hence, taking inequality (1.81) into account, we obtain

$$\begin{aligned} &\left|\langle\tilde{\beta}_{f_1}(t_1)\ \ldots\ \tilde{\beta}_{f_{s-1}}(t_{s-1})\,\beta_{f_s}(0)\rangle_\Gamma - \langle\beta_{f_1}(0)\ \ldots\ \beta_{f_{s-1}}(0)\,\beta_{f_s}(0)\rangle_\Gamma\right| \\ &\quad \leqslant \sum_{j=1}^{s-1}\ \sum_{q=1}^{j}\left|\int\limits_{t_j}^{t_{j+1}} \sqrt{\left\langle \tilde{\beta}_{f_1}(t)\ \ldots\ \frac{d\tilde{\beta}_{f_q}(t)}{dt}\ \frac{d\tilde{\beta}^+_{f_q}(t)}{dt}\ \ldots\ \tilde{\beta}^+_{f_1}(t)\right\rangle_\Gamma}\, dt\right|. \end{aligned} \tag{1.83}$$

We turn now to the equations of motion (1.20) and (1.22) for the Hamiltonian Γ and represent them in the form

$$\frac{d\beta_f(t)}{dt} + i\omega(f)\,\beta_f(t) = Z_f(t),$$

in which

$$\begin{aligned} Z_f &= -iR_f \quad \text{if} \quad \beta_f = \alpha_f, \\ Z_f &= iR_f^+ \quad \text{if} \quad \beta_f = \alpha_f^+. \end{aligned} \tag{1.84}$$

We therefore obtain for the functions (1.78):

$$\frac{d\tilde{\beta}_f(t)}{dt} = Z_f(t)e^{i\omega(f)t}. \tag{1.85}$$

Consequently, we have:

$$\begin{aligned} &\left\langle \tilde{\beta}_f(t)\ \ldots\ \frac{d\tilde{\beta}_{f_q}(t)}{dt}\,\frac{d\tilde{\beta}^+_{f_q}(t)}{dt}\ \ldots\ \tilde{\beta}^*_{f_1}(t)\right\rangle_\Gamma \\ &\quad = \langle\beta_{f_1}(t)\ \ldots\ Z_{f_q}(t)\,Z^+_{f_q}(t)\ \ldots\ \beta^+_{f_1}(t)\rangle_\Gamma \\ &\quad = \langle\beta_{f_1}(0)\ \ldots\ Z_{f_q}(0)\,Z^+_{f_q}(0)\ \ldots\ \beta^+_{f_1}(0)\rangle_\Gamma \\ &\quad = \langle\beta_{f_1}\ \ldots\ Z_{f_q}Z^+_{f_q}\ \ldots\ \beta^+_{f_1}\rangle_\Gamma . \end{aligned}$$

We now take our inequalities (1.37) and (1.38) into account. Since

$$Z_f Z_f^+ = R_f R_f^+ \quad \text{or} \quad R_f^+ R_f,$$

we convince ourselves that

$$\sqrt{\left\langle \tilde{\beta}_{f_1}(t)\ \ldots\ \frac{d\tilde{\beta}_{f_q}(t)}{dt}\ \frac{d\tilde{\beta}^+_{f_q}(t)}{dt}\ \ldots\ \tilde{\beta}_{f_1}(t)\right\rangle_\Gamma} \leqslant \eta_V^{(q)} \leqslant \eta_V^{(s)}.$$

Thus, from the inequality (1.83) we find:

$$\left|\langle\tilde{\beta}_{f_1}(t_1)\,\ldots\,\tilde{\beta}_{f_{s-1}}(t)\;\beta_{f_s}(0)\rangle_\Gamma-\langle\beta_{f_1}(0)\,\ldots\,\beta_{f_{s-1}}(0)\,\beta_{f_s}(0)\rangle_\Gamma\right|$$

$$\leqslant\sum_{j=1}^{s-1}\eta_V^{(s)}j\,|t_j-t_{j+1}|.$$

The relation (1.80) is now proved, and the proof of inequality (1.75) is thereby completed.

We see from inequality (1.75), in particular, that the convergence

$$\langle\beta_{f_1}(t_1)\,\ldots\,\beta_{f_s}(t_s)\rangle_\Gamma-\langle\beta_{f_1}(t_1)\,\ldots\,\beta_{f_s}(t_s)\rangle_{\Gamma_a}\to 0$$

as

$$V\to\infty$$

is uniform with respect to $t_1, t_2, \ldots, t_s$ in the region in which their differences are bounded:[†]

$$|t_j-t_{j+1}|\leqslant S.$$

(Here S is any fixed constant that is, of course, independent of V.)

Up to now we have considered averages composed of products of operators $\alpha_f(t')$, $\alpha_f(t'')$. We turn now to examine products of the "original", "old" Fermi operators a_f, a_f^+.

[†] To elucidate intuitively the reason for the absence of uniform convergence with respect to $t_1, t_2, \ldots, t_s$ in the *entire* space of the points $t_1, \ldots, t_s$, we shall consider the simple example when

$$\Gamma=\Gamma_a+\Gamma_1,$$

with

$$\Gamma_a=\sum_f E(f)\,\alpha_f^+\alpha_f,$$

$$\Gamma_1=\sum_f \varepsilon\alpha_f^+\alpha_f,\qquad \varepsilon>0\qquad(\varepsilon\to 0\quad\text{as}\quad V\to\infty).$$

We obtain, in this case,

$$\langle\alpha_f^+(t_1)\,\alpha_f(t_2)\rangle_\Gamma=\frac{1}{1+e^{(E(f)+\varepsilon)/\theta}}\,e^{i(E(f)+\varepsilon)(t_1-t_2)},$$

$$\langle\alpha_f^+(t_1)\,\alpha_f(t_2)\rangle_{\Gamma_a}=\frac{1}{1+e^{(E(f))/\theta}}\,e^{iE(f)(t_1-t_2)}.$$

It is clear from this that these averages can be close to each other only so long as $\varepsilon\,|t_1-t_2|$ is small.

We shall be concerned with averages of the type

$$\langle A_{f_1}(t_1)\,\dots\,A_{f_s}(t_s)\rangle_{\Gamma},\quad \langle A_{f_1}(t_1)\,\dots\,A_{f_s}(t_s)\rangle_{\Gamma_a}$$

where

$$A_f(t)=a_f(t)\quad\text{or}\quad a_f^+(t).$$

Here, the Fermi operators $A_f(t)$ occurring in the averages

$$\langle\dots\rangle_{\Gamma},\quad \langle\dots\rangle_{\Gamma_a}$$

are determined by the equations of motion for the Hamiltonians Γ and Γ_a respectively.

We note that, by virtue of the relations (1.18),

$$\begin{aligned} a_f(t) &= u(f)\,\alpha_f(t)-v(f)\,\alpha^+_{-f}(t),\\ a_f^+(t) &= u(f)\,\alpha_f^+(t)-v^*(f)\,\alpha_{-f}(t).\end{aligned}$$

We write these canonical u–v transformations in the abbreviated form:

$$A_f(t)=u_1(f)\,\beta_{f,1}(t)+u_2(f)\,\beta_{f,2}(t). \tag{1.86}$$

Here,

$$|u_1(f)|^2+|u_2(f)|^2=1,$$

and, therefore,

$$\sum_{\gamma=1,2}|u_\gamma(f)|\leqslant\sqrt{2}. \tag{1.87}$$

Using the relations (1.86), we obtain

$$\begin{aligned}&\langle A_{f_1}(t_1)\,\dots\,A_{f_s}(t_s)\rangle_{\Gamma}-\langle A_{f_1}(t_1)\,\dots\,A_{f_s}(t_s)\rangle_{\Gamma_a}\\ &=\prod_{j=1}^{s}\sum_{\gamma_j=1,2}u_{\gamma_j}(f_j)\left\{\langle\beta_{f_1\gamma_1}(t_1)\,\dots\,\beta_{f_s\gamma_s}(t_s)\rangle_{\Gamma}-\langle\beta_{f_1\gamma_1}(t_1)\,\dots\,\beta_{f_s\gamma_s}(t_s)\rangle_{\Gamma_a}\right\}.\end{aligned}$$

Hence, using (1.75) and (1.87), we can convince ourselves that

$$\begin{aligned}&|\langle A_{f_1}(t_1)\,\dots\,A_{f_s}(t_s)\rangle_{\Gamma}-\langle A_{f_1}(t_1)\,\dots\,A_{f_s}(t_s)\rangle_{\Gamma_a}|\\ &\leqslant\left\{\eta_V^{(s)}\sum_{j=1}^{s-1}j\,|t_j-t_{j+1}|+2^{(s/2)+1}\mathcal{K}\sqrt{\varepsilon_V+\frac{M_2}{4V}}\right\}\prod_{j=1}^{s}\sum_{\gamma_j=1,2}|u_{\gamma_j}(f_j)|\\ &\leqslant 2^{s/2}\left\{\eta_V^{(s)}\sum_{j=1}^{s-1}j\,|t_j-t_{j+1}|+2^{(s/2)+1}\mathcal{K}\sqrt{\varepsilon_V+\frac{M_2}{4V}}\right\}.\end{aligned}$$

Thus, we have proved the following theorem:

THEOREM II. *Under the conditions I, the following inequality holds:*

$$
\begin{aligned}
&|\langle A_{f_1}(t_1)\,\ldots\,A_{f_s}(t_s)\rangle_\Gamma-\langle A_{f_1}(t_1)\,\ldots\,A_{f_s}(t_s)\rangle_{\Gamma_a}|\leqslant\delta_V^{(s)}\\
&=2^{(s+1)/2}\left\{\frac{M_1M_2}{4V}+M_1\left(\sqrt{\varepsilon_V}+\frac{s-1}{V}\sqrt{M_1}\right)^2\right\}^{1/2}\sum_{j=1}^{s-1}j\,|t_j-t_{j+1}|\\
&+2^{s+1}\mathcal{K}\sqrt{\varepsilon_V+\frac{M_2}{4V}}. \qquad (1.88)
\end{aligned}
$$

Here we can always take the value (1.56) for $\mathcal{K}$. If

$$E(f_1)\geqslant\gamma,\ \ldots,\ E(f_s)\geqslant\gamma,$$

we can take for $\mathcal{K}$ the expression (1.57).

CHAPTER 2

CONSTRUCTION OF A PROOF OF THE GENERALIZED ASYMPTOTIC RELATIONS FOR THE MANY-TIME CORRELATION AVERAGES

IN THIS chapter we shall obtain generalized asymptotic relations for many-time correlation averages, T-products and Green's functions, starting, essentially, from the results of Chapter 1 and the theorems which were proved there. The systematic derivations for finding the generalized asymptotic relations are summarized by theorems.

§ 1. SELECTION RULES AND CALCULATION OF THE AVERAGES

We shall now study the "selection rules" for the averages considered in Chapter 1, § 8, i.e. for

$$\langle A_{f_1}(t_1) \ldots A_{f_s}(t_s)\rangle.$$

As has already been noted, for *odd s* these averages are identically zero.

We shall now formulate more stringent selection rules. We take the operators

$$n_{f_0} = \overset{+}{a}_{f_0} a_{f_0}, \quad n_{-f_0} = \overset{+}{a}_{-f_0} a_{-f_0},$$

and note that their difference

$$(n_{f_0} - n_{-f_0}) \tag{2.1}$$

commutes, for any value of f_0, with all operators of the form

$$a_f^+ a_f, \quad a_{-f} a_f, \quad a_f^+ a_{-f}^+.$$

Therefore, the operators (2.1) will commute with the model (Γ) and trial (Γ_a) Hamiltonians, and will consequently be integrals of motion for them.

We introduce the unitary operators

$$U_{f_0} = e^{i\varphi(n_{f_0} - n_{-f_0})}, \quad U_{f_0}^+ = e^{-i\varphi(n_{f_0} - n_{-f_0})},$$

where φ is an arbitrary real number. By virtue of the above arguments,

$$\Gamma U_{f_0} = U_{f_0} \Gamma$$

and, therefore,

$$e^{-\Gamma/\theta} U_{f_0} = U_{f_0} e^{-\Gamma/\theta}.$$

Consequently, we can write

$$\begin{aligned}
\langle A_{f_1}(t_1) \ldots A_{f_s}(t_s) \rangle_\Gamma &= \frac{\operatorname{Tr}\left\{A_{f_1}(t_1) \ldots A_{f_s}(t_s)\, e^{-\Gamma/\theta} U_{f_0} U_{f_0}^+\right\}}{\operatorname{Tr} e^{-\Gamma/\theta}} \\
&= \frac{\operatorname{Tr}\left\{U_{f_0}^+ A_{f_1}(t_1) \ \ldots \ A_{f_s}(t_s)\, e^{-\Gamma/\theta} U_{f_0}\right\}}{\operatorname{Tr} e^{-\Gamma/\theta}} \\
&= \frac{\operatorname{Tr}\left\{U_{f_0}^+ A_{f_1}(t_1) \ \ldots \ A_{f_s}(t_s)\, U_{f_0} e^{-\Gamma/\theta}\right\}}{\operatorname{Tr} e^{-\Gamma/\theta}} \\
&= \left\langle U_{f_0}^+ A_{f_1}(t_1) \ \ldots \ A_{f_s}(t_s)\, U_{f_0} \right\rangle_\Gamma .
\end{aligned}$$

But in view of the unitarity, we have, identically,

$$\begin{aligned}
&\left\langle U_{f_0}^+ A_{f_1}(t_1) \ldots A_{f_s}(t_s)\, U_{f_0} \right\rangle_\Gamma \\
&\quad = \left\langle \left\{U_{f_0}^+ A_{f_1}(t_1)\, U_{f_0}\right\} \left\{U_{f_0}^+ A_{f_2}(t_2)\, U_{f_0}\right\} \ \ldots \ \left\{U_{f_0}^+ A_{f_s}(t_s)\, U_{f_0}\right\} \right\rangle_\Gamma .
\end{aligned}$$

Thus, we have proved that, for any value of f_0:

$$\begin{aligned}
&\langle A_{f_1}(t_1) \ldots A_{f_s}(t_s) \rangle_\Gamma \\
&\quad = \left\langle \left\{U_{f_0}^+ A_{f_1}(t_1)\, U_{f_0}\right\} \ldots \left\{U_{f_0}^+ A_{f_s}(t_s)\, U_{f_0}\right\} \right\rangle_\Gamma . \qquad (2.2)
\end{aligned}$$

We note now that

$$\begin{aligned} U_{f_0}^{+} a_{f_0} U_{f_0} &= e^{i\varphi} a_{f_0}, & U_{f_0}^{+} a_{-f_0} U_{f_0} &= e^{-i\varphi} a_{-f_0} \\ U_{f_0}^{+} a_{-f_0}^{+} U_{f_0} &= e^{i\varphi} a_{-f_0}^{+}, & U_{f_0}^{+} a_{f_0}^{+} U_{f_0} &= e^{-i\varphi} a_{f_0}^{+}, \\ U_{f_0}^{+} a_f U_{f_0} &= a_f & \text{if} \quad f &\neq f_0,\ -f_0, \\ U_{f_0}^{+} a_f^{+} U_{f_0} &= a_f^{+} & \text{if} \quad f &\neq f_0,\ -f_0. \end{aligned} \tag{2.3}$$

Since the operator $n_{f_0} - n_{-f_0}$ and, consequently, also the operators U_{f_0} and $U_{f_0}^{+}$ are integrals of motion for the Hamiltonian Γ, the relations (2.3) will remain valid if in them we replace the operators a_f and a_f^{+} respectively by the operator functions of time $a_f(t)$ and $a_f^{+}(t)$, which are determined by the equations of motion for the Hamiltonian Γ.

Having noted this, we turn to the equality (2.2). We denote by $N_{+}(f_0)$ the number of operators equal to a_{f_0} or $a_{-f_0}^{+}$ amongst the operators $A_{f_1}, \ldots, A_{f_s}$, and by $N_{-}(f_0)$ the number of these operators equal to $a_{f_0}^{+}$ or a_{-f_0}. The right-hand side of (2.2) is then equal to

$$e^{i\{N_{+}(f_0) - N_{-}(f_0)\}\varphi} \langle A_{f_1}(t_1) \ldots A_{f_s}(t_s) \rangle_{\Gamma}$$

and we obtain

$$(1 - e^{i\{N_{+}(f_0) - N_{-}(f_0)\}\varphi}) \langle A_{f_1}(t_1) \ldots A_{f_s}(t_s) \rangle_{\Gamma} = 0.$$

Hence, in view of the arbitrariness of the phase φ, it follows that

$$\langle A_{f_1}(t_1) \ldots A_{f_s}(t_s) \rangle = 0$$

if the following inequality holds for any index f_0:

$$N_{+}(f_0) - N_{-}(f_0) \neq 0. \tag{2.4}$$

Two operators A_{f_j} and A_{f_q} from the operators $A_{f_1}, \ldots, A_{f_s}$ will be said to form a pair if

1. for $A_{f_j} = a_{f_j}$, $A_{f_q} = a_{f_j}^{+}$ or a_{-f_j};
2. for $A_{f_j} = a_{f_j}^{+}$, $A_{f_q} = a_{f_j}$ or $a_{-f_j}^{+}$.

It can be seen that if the operators A_{f_j} and A_{f_q} form a pair, then $f_q = \pm f_j$.

Let there be among the operators $A_{f_1}, \ldots, A_{f_s}$ an operator A_{f_j} for which it is impossible to select an operator A_{f_q} forming a pair with it. In this case, we shall say that A_{f_j} is an unpaired operator.

We shall show now that if among the operators $A_{f_1}, \ldots, A_{f_s}$ there is at least one unpaired operator, then

$$\langle A_{f_1}(t_1) \ldots A_{f_s}(t_s)\rangle_\Gamma = 0.$$

In fact, let A_{f_j} be such an unpaired operator. There are two possibilities:

$$1. \qquad A_{f_j} = a_{f_j};$$

$$2. \qquad A_{f_j} = a^+_{f_j}.$$

In the first case, none of the operators $A_{f_1}, \ldots, A_{f_s}$ is equal to $a^+_{f_j}$ or a_{-f_j}. In view of this,

$$N_-(f_j) = 0.$$

On the other hand,

$$N_+(f_j) \geqslant 1, \quad \text{and} \quad N_+(f_j) - N_-(f_j) \geqslant 1.$$

In the second case,

$$N_-(f_j) \geqslant 1, \quad N_+(f_j) = 0, \quad \text{and} \quad N_+(f_j) - N_-(f_j) \leqslant -1.$$

Thus, by virtue of (2.4), our assertion is proved. We see, consequently, that the average

$$\langle A_{f_1}(t_1) \ldots A_{f_s}(t_s)\rangle_\Gamma$$

can be non-zero only if all the operators $A_{f_1}, A_{f_2}, \ldots, A_{f_s}$ can be grouped in pairs.

We have just considered averages for the Hamiltonian Γ. From the arguments given, it is clear that the results obtained are also valid for averages taken over the Hamiltonian Γ_a. We further note that, since the trial Hamiltonian Γ_a is quadratic in the Fermi operators, in calculating expressions

$$\langle A_{f_1}(t_1) \ldots A_{f_s}(t_s)\rangle_{\Gamma_a}, \tag{2.5}$$

we can make use of the generalized Wick procedure established by Bloch and de Dominicis (1958, 1959). In accordance with this procedure, the average (2.5) is equal to a sum of products of binary averages:

$$\sum \eta \prod \langle A_{f_j}(t_j)\, A_{f_q}(t_q)\rangle_{\Gamma_s}; \qquad \eta = \pm 1, \quad f_q = \pm f_j. \tag{2.6}$$

The summation here is performed over all possible pairings in the product $A_{f_1} \ldots A_{f_s}$. The number of pairs in the sum (2.6) is equal, consequently, to the number of ways of combining the operators $A_{f_1}, \ldots, A_{f_s}$ in pairs. This number is obviously less than

$$1.\,3 \ldots (s-3)(s-1), \qquad s \text{ even}, s \geqslant 2.$$

The binary averages occurring in (2.6) are calculated directly by means of the transformations (1.18).

We have the following expressions for the binary averages:

$$\begin{aligned}
\langle a_f(t)\, a_{-f}(\tau)\rangle_{\Gamma_s} &= \frac{\Lambda(f)}{2E(f)}\left\{\frac{e^{iE(f)(t-\tau)}}{1+e^{\{E(f)\}/\theta}} - \frac{e^{-iE(f)(t-\tau)}}{1+e^{-\{E(f)\}/\theta}}\right\},\\
\langle a_f(t)\, a_f^{+}(\tau)\rangle_{\Gamma_s} &= \frac{1}{2}\left(1-\frac{T(f)}{E(f)}\right)\frac{e^{iE(f)(t-\tau)}}{1+e^{E(f)/\theta}}\\
&\quad + \frac{1}{2}\left(1+\frac{T(f)}{E(f)}\right)\frac{e^{-iE(f)(t-\tau)}}{1+e^{-\{E(f)\}/\theta}},\\
\langle a_f^{+}(t)\, a_f(\tau)\rangle_{\Gamma_s} &= \frac{1}{2}\left(1+\frac{T(f)}{E(f)}\right)\frac{e^{iE(f)(t-\tau)}}{1+e^{\{E(f)\}/\theta}}\\
&\quad + \frac{1}{2}\left(1-\frac{T(f)}{E(f)}\right)\frac{e^{-iE(f)(t-\tau)}}{1+e^{-\{E(f)\}/\theta}},\\
\langle a_f^{+}(t)\, a_{-f}^{+}(\tau)\rangle_{\Gamma_s} &= -\frac{\Lambda(f)}{2E(f)}\left\{\frac{e^{iE(f)(t-\tau)}}{1+e^{\{E(f)\}/\theta}} - \frac{e^{-iE(f)(t-\tau)}}{1+e^{-\{E(f)\}/\theta}}\right\}.
\end{aligned} \tag{2.7}$$

Thus, the average (2.5) is a polynomial in the expressions (2.7), in which we replace f by f_j, and t and τ by t_j and t_q.

This polynomial

$$\mathcal{P}(f_1, \ldots, f_s;\, t_1, \ldots, t_s), \tag{2.8}$$

is defined in the whole space $\phi^{(s)}$ (i.e. for all points $(f_1, \ldots, f_s)$), since the functions

$$\varDelta(f), \quad T(f), \quad E(f)$$

are defined everywhere in $\phi_V^{(s)}$.

If the function $\varDelta(f)$ does not depend on V, the above polynomial is also independent of V.

On the other hand, the equality

$$\langle A_{f_1}(t_1) \ldots A_{f_s}(t_s)\rangle_{\Gamma_a} = \mathcal{D}(f_1, \ldots, f_s; \; t_1, \ldots, t_s). \tag{2.9}$$

holds, obviously, only for the quasi-discrete indices

$$f = (p, \sigma), \quad p = \left(\frac{2\pi n_1}{L}, \frac{2\pi n_2}{L}, \frac{2\pi n_3}{L}\right),$$

i.e. only on the set $\phi_V^{(s)}$, since only for these indices (f) do the actual operators

$$A_f = a_f, a_f^+$$

have any meaning. Thus, the dependence on V of the above averages over the Hamiltonian Γ_a (in the case when $\varDelta(f)$ is independent of V) is due entirely to the fact that for $(f_1, \ldots, f_s)$ we take the points of the set $\phi_V^{(s)}$.

§ 2. GENERALIZED CONVERGENCE

We introduce the concept of convergence for functions

$$F_V(f_1, \ldots, f_s) \quad (V \to \infty)$$

defined on the sets $\phi_V^{(s)}$. The function $F_V(f_1, \ldots, f_s)$ will be said to converge uniformly (as $V \to \infty$) to some function $F(f_1, \ldots, f_s)$ defined in the whole $\phi^{(s)}$, if one can find a sequence of numbers

$$\delta_V \to 0, \quad V \to \infty$$

such that the inequality

$$|F_V(f_1, \ldots, f_s) - F(f_1, \ldots, f_s)| \leq \delta_V$$

is fulfilled everywhere on the sets $\phi_V^{(s)}$. We note now that, by virtue of (1.88) and (2.9), we have

$$
\begin{aligned}
&|\langle A_{f_1}(t_1)\ldots A_{f_s}(t_s)\rangle_\Gamma - \mathcal{P}(f_1,\ldots,f_s;\, t_1,\ldots,t_s)|\\
&\leqslant 2^{(s+1)/2}\left\{\frac{M_1M_2}{4V}+M_1\left(\sqrt{\varepsilon_V}+\frac{s-1}{V}\sqrt{M_1}\right)^2\right\}^{1/2}\sum_{j=1}^{s-1} j\,|t_j-t_{j+1}| \qquad (2.10)\\
&\quad +2^{s+1}\mathcal{K}\sqrt{\varepsilon_V+\frac{M_2}{4V}}
\end{aligned}
$$

everywhere on $\phi_V^{(s)}$.

Thus, in accordance with the definition of convergence introduced above, the following rider to Theorem II can be seen to be valid. *If the conditions I are satisfied and, in addition, the function $A(f)$ is independent of V, then the limit*

$$\lim_{V\to\infty}\langle A_{f_1}(t_1)\ldots A_{f_s}(t_s)\rangle_\Gamma = \mathcal{P}(f_1,\ldots,f_s;\, t_1,\ldots,t_s) \qquad (2.11)$$

exists.

Up to now we have considered averages composed of products of Fermi operators a_f, a_f^+. We pass on now to the operator field functions

$$
\begin{aligned}
\psi_\sigma(\boldsymbol{r}, t) &= \frac{1}{\sqrt{V}}\sum_p a_{p,\sigma}(t)e^{i(\boldsymbol{p}\cdot\boldsymbol{r})},\\
\psi_\sigma^+(\boldsymbol{r}, t) &= \frac{1}{\sqrt{V}}\sum_p a_{p,\sigma}^+(t)e^{-i(\boldsymbol{p}\cdot\boldsymbol{r})}.
\end{aligned}
\qquad (2.12)
$$

Here, the summation runs, as before, over the set of quasi-discrete p.

We shall be concerned with averages composed of a product of operator field functions, constructed on the basis of model and trial Hamiltonians, Γ and Γ_a,

$$\langle\varphi_{\sigma_1}(\boldsymbol{r}_1, t_1)\ldots\varphi_{\sigma_s}(\boldsymbol{r}_s, t_s)\rangle_{\Gamma,\Gamma_a},$$

where s is an even number (for odd s the averages are identically zero), and where

$$\varphi_\sigma(\boldsymbol{r}, t) = \psi_\sigma(\boldsymbol{r}, t) \quad \text{or} \quad \psi_\sigma^+(\boldsymbol{r}, t).$$

By virtue of (2.12), we can write

$$\varphi_\sigma(\boldsymbol{r}, t) = \frac{1}{\sqrt{V}} \sum_{\boldsymbol{p}} A_{\boldsymbol{p},\sigma}(t) e^{i\varepsilon(\boldsymbol{p}\cdot\boldsymbol{r})},$$

where

$$A_{\boldsymbol{p},\sigma} = a_{\boldsymbol{p},\sigma}, \quad \varepsilon = 1, \quad \text{if} \quad \varphi = \psi,$$

and

$$A_{\boldsymbol{p},\sigma} = a^+_{\boldsymbol{p},\sigma}, \quad \varepsilon = -1, \quad \text{if} \quad \varphi = \psi^+ .$$

Thus,

$$\begin{aligned} &\langle \varphi_{\sigma_1}(\boldsymbol{r}_1, t_1) \ldots \varphi_{\sigma_s}(\boldsymbol{r}_s, t_s) \rangle_{\Gamma, \Gamma_a} \\ &= \frac{1}{V^{s/2}} \sum_{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_s} \langle A_{\boldsymbol{p}_1\sigma_1}(t_1) \ldots A_{\boldsymbol{p}_s\sigma_s}(t_s) \rangle_{\Gamma, \Gamma_a} e^{i\{\varepsilon_1(\boldsymbol{p}_1\cdot\boldsymbol{r}_1) + \ldots + \varepsilon_s(\boldsymbol{p}_s\cdot\boldsymbol{r}_s)\}}. \end{aligned} \tag{2.13}$$

We shall consider the difference

$$\langle \varphi_{\sigma_1}(\boldsymbol{r}_1, t_1) \ldots \varphi_{\sigma_s}(\boldsymbol{r}_s, t_s) \rangle_{\Gamma} - \langle \varphi_{\sigma_1}(\boldsymbol{r}_1, t_1) \ldots \varphi_{\sigma_s}(\boldsymbol{r}_s, t_s) \rangle_{\Gamma_a},$$

and show that for $V \to \infty$ it tends to zero, though now in the *generalized sense* customary in the theory of generalized functions. We recall here that in this theory,

$$\begin{gathered} F_V(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_s) \to F(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_s), \\ F(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_s) = \lim_{(V\to\infty)} F_V(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_s) \end{gathered} \tag{2.14}$$

is called generalized convergence. We shall consider the class $C(q, \nu)$ (where q and ν are positive numbers) of continuous and continuously differentiable functions $h(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_s)$ such that

$$\{|r_1| + \ldots + |r_s|\}^j \, |h(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_s)| \leqslant \text{const}$$

$$(j = 0, 1, 2, \ldots, \nu),$$

and

$$\{|r_1| + \ldots + |r_s|\}^j \left| \frac{\partial^{q_1 + \ldots + q_{3s}} h}{\partial x_{1,\alpha_1}^{q_1} \ldots \partial x_{3s,\alpha_{3s}}^{q_{3s}}} \right| \leqslant \text{const}$$

$$(j = 0, 1, 2, \ldots, \nu; \qquad q_1 + \ldots + q_{3s} = 0, 1, \ldots, q; \qquad \alpha_l = 1, 2, 3)$$

in the whole space $E^{(s)}$ of points $(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_s)$.

Here,

$$(x_{l,1}, x_{l,2}, x_{l,3}) = \boldsymbol{r}_l .$$

Then, if we can fix the positive numbers q and ν in such a way that the relation

$$\int h(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_s) F_V(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_s)\, d\boldsymbol{r}_1 \ldots d\boldsymbol{r}_s$$
$$\to \int h(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_s) F(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_s) d\boldsymbol{r}_1 \ldots d\boldsymbol{r}_s$$
$$(V \to \infty)$$

is valid for any function $h(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_s)$ from the class $C(q, \nu)$, there is said to be generalized convergence (2.14). For our purposes, we shall confine ourselves to examining the class $\mathcal{L} = C(q_1, \nu_1)$ in which the numbers q_1 and ν_1 are chosen in such a way that the Fourier transform of the function $h(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_s)$

$$\tilde{h}(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_s) = \int h(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_s) e^{i(\boldsymbol{p}_1\cdot\boldsymbol{r}_1 + \cdots + \boldsymbol{p}_s\cdot\boldsymbol{r}_s)}\, d\boldsymbol{r}_1 \ldots d\boldsymbol{r}_s$$

is a continuous function of $(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_s)$ in the whole space $E^{(s)}$, and satisfies the inequality

$$|\tilde{h}(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_s)| \leqslant \frac{\mathcal{K}_h}{\prod_{j=1}^{s} (|\boldsymbol{p}_j|^2 + M_h)}, \tag{2.15}$$
$$\mathcal{K}_h = \text{const} > 0,$$
$$M_h = \text{const} > 0.$$

We shall show that for any function $h(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_s)$ in this class $\mathcal{L}$

$$\int h(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_s) \langle \varphi_{\sigma_1}(\boldsymbol{r}_1, t_1) \ldots \varphi_{\sigma_s}(\boldsymbol{r}_s, t_s)\rangle_\Gamma\, d\boldsymbol{r}_1 \ldots d\boldsymbol{r}_s$$
$$- \int h(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_s) \langle \varphi_{\sigma_1}(\boldsymbol{r}_1, t_1) \ldots \varphi_{\sigma_s}(\boldsymbol{r}_s, t_s)\rangle_{\Gamma_a}\, d\boldsymbol{r}_1 \ldots d\boldsymbol{r}_s \to 0$$
$$\text{as} \quad V \to \infty.$$

Re-expressing this using (2.13), we shall prove that

$$\frac{1}{V^{s/2}} \sum_{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_s} \tilde{h}(\varepsilon_1 \boldsymbol{p}_1, \ldots, \varepsilon_s \boldsymbol{p}_s) \{\langle A_{\boldsymbol{p}_1\sigma_1}(t_1) \ldots A_{\boldsymbol{p}_s\sigma_s}(t_s)\rangle_\Gamma - \langle A_{\boldsymbol{p}_1\sigma_1}(t_1) \ldots A_{\boldsymbol{p}_s\sigma_s}(t_s)\rangle_{\Gamma_a}\} \to 0. \tag{2.16}$$

Here the summation runs over the points $(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_s)$ of the quasi-discrete set $E_V^{(s)}$. We shall thereby also prove that, in the generalized sense

$$\begin{aligned}&\langle \varphi_{\sigma_1}(\boldsymbol{r}_1, t_1) \ldots \varphi_{\sigma_s}(\boldsymbol{r}_s, t_s)\rangle_\Gamma \\ &\quad -\langle \varphi_{\sigma_1}(\boldsymbol{r}_1, t_1) \ldots \varphi_{\sigma_s}(\boldsymbol{r}_s, t_s)\rangle_{\Gamma a} \to 0 \quad \text{as} \quad V \to \infty.\end{aligned} \tag{2.17}$$

§ 3. REMARK

The question may arise as to why in considering averages of products of operator field functions we have had to confine ourselves to proving only the generalized convergence (2.17).

The point is that, even at fixed finite volume V, the sums (2.13) representing the actual averages can, generally speaking, be divergent in the usual sense.

We take as an example the expression

$$\langle \psi_\sigma(\boldsymbol{r}_1, t_1)\, \psi_\sigma^+(\boldsymbol{r}_2, t_2)\rangle_{\Gamma_a}.$$

We have

$$\langle \psi_\sigma(\boldsymbol{r}_1, t_1)\, \psi_\sigma^+(\boldsymbol{r}_2, t_2)\rangle_{\Gamma_a} = \frac{1}{V}\sum_{p_1, p_2} \langle a_{p_1\sigma}(t_1)\, a_{p_2\sigma}^+(t_2)\rangle_{\Gamma_a} e^{i\{(p_1\cdot r_1)-(p_2\cdot r_2)\}}.$$

Hence, taking into account the formulae (2.7) and the selection rules established earlier, we obtain

$$\begin{aligned}\langle \psi_\sigma(\boldsymbol{r}_1, t_1)\, \psi_\sigma^+(\boldsymbol{r}_2, t_2)\rangle_{\Gamma_a} &= \frac{1}{V}\sum_p \langle a_{p,\sigma}(t_1)\, a_{p,\sigma}^+(t_2)\rangle_{\Gamma_a} e^{i(p\cdot r_1 - r_2)} \\ &= \frac{1}{2V}\sum_p \left(1-\frac{T(f)}{E(f)}\right)\frac{e^{iE(f)(t_1-t_2)}}{1+e^{\{E(f)\}/\theta}}\, e^{i(p\cdot r_1 - r_2)} \\ &\quad + \frac{1}{2V}\sum_p \left(1+\frac{T(f)}{E(f)}\right)\frac{e^{-iE(f)(t_1-t_2)}}{1+e^{-\{E(f)\}/\theta}}\, e^{i(p\cdot r_1 - r_2)}.\end{aligned}$$

For simplicity, we take the case when

$$\Delta(f) = 0$$

identically. Then,

$$E(f) = |T(f)| = \left| \frac{p^2}{2m} - \mu \right|$$

and we have

$$\langle \psi_\sigma(\boldsymbol{r}_1, t_1)\, \psi_\sigma^+(\boldsymbol{r}_2, t_2) \rangle_{\Gamma_a}$$

$$= \frac{1}{V} \sum_{\{(p^2/2m)<\mu\}} \frac{e^{i\,|(p^2/2m)-\mu|(t_1-t_2)}}{1+e^{|(p^2/2m)-\mu|/\theta}}\, e^{i(\boldsymbol{p}\cdot\boldsymbol{r}_1-\boldsymbol{r}_2)}$$
$$+\frac{1}{V} \sum_{\{(p^2/2m)>\mu\}} \frac{e^{-i\{(p^2/2m)-\mu\}(t_1-t_2)}}{1+e^{-\{(p^2/2m)-\mu\}/\theta}}\, e^{i(\boldsymbol{p}\cdot\boldsymbol{r}_1-\boldsymbol{r}_2)}. \tag{2.18}$$

It can be seen that for fixed V the first sum contains a finite number of terms and the second sum an infinite number. In this case,

$$\frac{1}{1+e^{-\{(p^2/2m)-\mu\}/\theta}} \to 1,$$

$$(|\boldsymbol{p}| \to \infty),$$

Consequently, the second sum is a divergent series (in the usual sense).

It is not difficult to see, in passing, that the given series is convergent in the generalized sense. In fact, multiplying it by a function $h(\boldsymbol{r}_1, \boldsymbol{r}_2)$ from the class $\mathscr{L}$ and integrating, we obtain the sum

$$\frac{1}{V} \sum_{\{(p^2/2m)\geqslant\mu\}} \frac{e^{-i\{(p^2/2m)-\mu\}(t_1-t_2)}}{1+e^{-\{(p^2/2m)-\mu\}/\theta}}\, \tilde{h}(\boldsymbol{p}, -\boldsymbol{p}),$$

which converges absolutely, since

$$\frac{1}{V} \sum_{\{(p^2/2m)\geqslant\mu\}} \frac{1}{1+e^{-\{(p^2/2m)-\mu\}/\theta}}\, |\tilde{h}(\boldsymbol{p}, -\boldsymbol{p})|$$

$$\leqslant \frac{\mathscr{K}_h}{V} \sum_{\{(p^2/2m)\geqslant\mu\}} \frac{1}{(p^2+M_h)^2} = \mathscr{K}_h \left\{ \int\limits_{\{(p^2/2m)\geqslant\mu\}} \frac{d\boldsymbol{p}}{(p^2+M_h)^2} + \zeta_V \right\},$$

where

$$\zeta_V \to 0 \quad \text{as} \quad V \to \infty.$$

Thus, even in this very simple case, we see that the average

$$\langle \psi_\sigma(\mathbf{r}_1, t_1)\,\psi_\sigma^+(\mathbf{r}_2, t_2)\rangle_{\Gamma_a} \tag{2.19}$$

is given by a sum (2.13) which is convergent only in the generalized sense; the expression (2.19) is, therefore, defined only as a generalized function of $\mathbf{r}_1$ and $\mathbf{r}_2$, even for finite V.

§ 4. PROOF OF THE ASYMPTOTIC RELATIONS

We turn now to the proof of the asymptotic relations (2.16). Using the notation

$$\mathcal{D}_V(h) = \frac{1}{V^{s/2}} \sum_{\mathbf{p}_1, \dots, \mathbf{p}_s} \tilde{h}(\varepsilon_1 \mathbf{p}_1, \dots, \varepsilon_s \mathbf{p}_s)\,\mathcal{D}_{p_1\sigma_1; \dots ; p_s\sigma_s},$$

$$\mathcal{D}_{p_1\sigma_1; \dots ; p_s\sigma_s} = \langle A_{p_1\sigma_1}(t_1) \dots A_{p_s\sigma_s}(t_s)\rangle_\Gamma - \langle A_{p_1\sigma_1}(t_1) \dots A_{p_s\sigma_s}(t_s)\rangle_{\Gamma_a}, \tag{2.20}$$

we obtain, by virtue of (2.15),

$$|\mathcal{D}_V(h)| \leq \frac{\mathcal{K}_h}{V^{s/2}} \sum_{p_1, \dots, p_s} |\mathcal{D}_{p_1\sigma_1; \dots ; p_s\sigma_s}| \prod_{j=1}^{s} \frac{1}{(p_j^2 + M_h)}. \tag{2.21}$$

We shall take into account here the above-mentioned selection rules and use them for $\mathcal{D}_{p_1\sigma_1; \dots ; p_s\sigma_s}$. Owing to these selection rules, the summation in inequality (2.21) need be carried out only over those $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_s$ for which the system of operators $A_{p_1\sigma_1}, \dots, A_{p_s\sigma_s}$ can be divided into pairs $A_{p_j\sigma_j}$, $A_{p_k\sigma_k}$ such that

$$\mathbf{p}_k = \mathbf{p}_j \quad \text{if} \quad \sigma_j = \sigma_k$$

and

$$\mathbf{p}_k = -\mathbf{p}_j \quad \text{if} \quad \sigma_j = -\sigma_k.$$

We denote

$$\mathbf{p}_{j_1} = \mathbf{q}_1, \quad \mathbf{p}_{j_2} = \mathbf{q}_2, \dots, \quad \mathbf{p}_{j_{s/2}} = \mathbf{q}_{s/2}.$$

We have

$$\prod_{j=1}^{s} \frac{1}{(p_j^2 + M_h)} = \prod_{j=1}^{s/2} \frac{1}{(q_j^2 + M_h)^2}.$$

Further, we denote the number of ways the operators $A_{p_1\sigma_1}, \dots, A_{p_s\sigma_s}$

can be divided into pairs by $l(s)$. Obviously,

$$l(s) < 1\cdot 3 \ldots (s-3)(s-1).$$

Then in the right-hand side of (2.21), the sum

$$\sum_{\mathbf{p}_1, \ldots, \mathbf{p}_s} (\ldots)$$

can be broken down into $l(s)$ sums of the type

$$\sum_{\mathbf{q}_1, \ldots, \mathbf{q}_{s/2}} (\ldots), \tag{2.22}$$

We note that the same terms can be found in different sums (2.22). To exclude these from certain of the sums (2.22), we must not perform the summation over all the $(\mathbf{q}_1, \ldots, \mathbf{q}_{s/2})$ of the quasi-discrete set under consideration; a number of "exclusions" of the type

$$(\mathbf{q}_l \neq \pm \mathbf{q}_{l'})$$

must be introduced.†

---

† We take, for example, an average of the form

$$\langle \psi_\sigma^+(\mathbf{r}_1, t_1)\, \psi_\sigma^+(\mathbf{r}_2, t_2)\, \psi_\sigma(\mathbf{r}_3, t_3)\, \psi_\sigma(\mathbf{r}_4, t_4)\rangle.$$

In this case we shall have $s = 4$ in (2.21), and

$$\begin{aligned}\mathcal{D}_{\mathbf{p}_1\sigma; \ldots; \mathbf{p}_4\sigma} = {} & \langle a^+_{\mathbf{p}_1\sigma}(t_1)\, a^+_{\mathbf{p}_2\sigma}(t_2)\, a_{\mathbf{p}_3\sigma}(t_3)\, a_{\mathbf{p}_4\sigma}(t_4)\rangle_\Gamma \\ & - \langle a^+_{\mathbf{p}_1\sigma}(t_1)\, a^+_{\mathbf{p}_2\sigma}(t_2)\, a_{\mathbf{p}_3\sigma}(t_3)\, a_{\mathbf{p}_4\sigma}(t_4)\rangle_{\Gamma_a}.\end{aligned}$$

From the selection rules, $D_{\mathbf{p}_1\sigma; \ldots; \mathbf{p}_4\sigma}$ can be non-zero here only if

$$\mathbf{p}_1 = \mathbf{p}_4, \quad \mathbf{p}_2 = \mathbf{p}_3$$

or

$$\mathbf{p}_1 = \mathbf{p}_3, \quad \mathbf{p}_2 = \mathbf{p}_4,$$

so that $l = 2$. As stipulated above, we denote:

$$\mathbf{p}_1 = \mathbf{q}_1, \quad \mathbf{p}_2 = \mathbf{q}_2.$$

We have

$$\begin{aligned}\sum_{\mathbf{p}_1, \ldots, \mathbf{p}_4} |\mathcal{D}_{\mathbf{p}_1\sigma; \ldots; \mathbf{p}_4\sigma}| \prod_{j=1}^{4} \frac{1}{(p_j^2 + M_\lambda)} & = \sum_{\mathbf{q}_1, \mathbf{q}_2} |\mathcal{D}_{\mathbf{q}_1\sigma; \mathbf{q}_2\sigma; \mathbf{q}_1\sigma; \mathbf{q}_2\sigma}| \prod_{j=1}^{2} \frac{1}{(q_j^2 + M_\lambda)^2} \\ & + \sum_{\substack{\mathbf{q}_1, \mathbf{q}_2 \\ (\mathbf{q}_1 \neq \mathbf{q}_2)}} |\mathcal{D}_{\mathbf{q}_1\sigma: \mathbf{q}_2\sigma; \mathbf{q}_2\sigma; \mathbf{q}_1\sigma}| \prod_{j=1}^{2} \frac{1}{(q_j^2 + M_\lambda)^2},\end{aligned}$$

since in the second sum on the right-hand side we must exclude the term $\mathbf{q}_1 = \mathbf{q}_2$, which is already accounted for in the first sum.

Since all the terms in the sums (2.22) being considered are positive, removing these exclusions only increases the upper bound. Thus, using (1.88), we find from (2.21) that

$$|\mathcal{D}_V(h)| \leqslant \delta_V^{(s)} \mathcal{K}_h l(s) \left\{ \frac{1}{V} \sum_q \frac{1}{(q^2+M_h)^2} \right\}^{s/2}. \tag{2.23}$$

But, on the other hand, it is not difficult to see that

$$\frac{1}{V} \sum_q \frac{1}{(q^2+M_h)^2} = \frac{1}{(2\pi)^3} \int \frac{dq}{(q^2+M_h)^2} + \zeta_V$$

$$= \frac{4\pi}{(2\pi)^3} \int_0^\infty \frac{R^2\,dR}{(R^2+M_h)^2} + \zeta_V,$$

where

$$\zeta_V \to 0 \quad \text{as} \quad V \to \infty.$$

Therefore we can write

$$\frac{1}{V} \sum_q \frac{1}{(q^2+M_h)^2} \leqslant C(M_h),$$

where $C(M)$ is a constant depending only on M. We have now from (2.23)

$$|\mathcal{D}_V(h)| \leqslant \delta_V^{(s)} \mathcal{K}_h l(s)\, C^{s/2}(M_h).$$

Thus, we have proved that the inequality:

$$\begin{aligned} &\left| \int h(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_s) \left\{ \langle \varphi_{\sigma_1}(\boldsymbol{r}_1, t_1) \ldots \varphi_{\sigma_s}(\boldsymbol{r}_s, t_s) \rangle_\Gamma \right.\right. \\ &\quad \left.\left. - \langle \varphi_{\sigma_1}(\boldsymbol{r}_1, t_1) \ldots \varphi_{\sigma_s}(\boldsymbol{r}_s, t_s) \rangle_{\Gamma_a} \right\} d\boldsymbol{r}_1 \ldots d\boldsymbol{r}_s \right| = |\mathcal{D}_V(h)| \\ &\leqslant C(s, \mathcal{K}_h, M_h) \left\{ \left(\frac{M_1 M_2}{4V} + M_1 \left[\sqrt{\varepsilon_V} + \frac{s-1}{V} \sqrt{M_1} \right]^2 \right)^{1/2} \right. \\ &\quad \left. \times \sum_{j=1}^{s-1} j\, |t_j - t_{j+1}| + 2^{(s+1)/2} \mathcal{K} \sqrt{\varepsilon_V + \frac{M_2}{4V}} \right\} \end{aligned} \tag{2.24}$$

is true for any function $h(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_s)$ of the class $\mathcal{L}$. Here,

$$C(s, \mathcal{K}_h, M_h) = 2^{(s+1)/2} l(s) \mathcal{K}_h C^{s/2}(M_h)$$

is a constant, depending only on s, $\mathcal{K}_h$ and M_h. The relations (2.24) also demonstrate the validity of the asymptotic relation (2.17).

We shall make one more remark with regard to the possibility of obtaining upper bounds which are uniform in the temperature interval $(0 < \theta \leqslant \theta_0)$ even when the condition $E(f) \geqslant \gamma$ is not fulfilled.

§ 5. REMARK ON THE CONSTRUCTION OF UNIFORM BOUNDS

As was remarked earlier, for $\mathcal{K}$ we can always take the expression (1.56):

$$\mathcal{K} = \frac{2}{\theta}\sqrt{M_1}.$$

In the case when, in the whole space ϕ,

$$E(f) \geqslant \gamma, \tag{2.25}$$

where γ is a positive non-zero constant, we can put (cf. (1.57))

$$\mathcal{K} = \frac{Q}{\gamma},$$

where

$$Q = \frac{3}{2}\left\{2M_1 + 2\left(\frac{2}{e}\right)^2 M_1\right\}^{1/2},$$

in the inequality (2.24). Thus, in this case, if $\varepsilon_V \to 0$ $(V \to \infty)$ uniformly in the temperature interval $(0 < \theta \leqslant \theta_0)$, our bound (2.24) will also be uniform in this interval. We shall show that this is so even if the condition (2.25) is not fulfilled.

In this case, we shall slightly modify the inequality (1.88), which we used to give explicit form to the relations (2.21). We fix some $\gamma > 0$. Then (1.88) is true for $\mathcal{K} = Q/\gamma$ in the region in which

$$|T(p_1)| \geqslant \gamma, \ \ldots, \ |T(p_s)| \geqslant \gamma,$$

since it follows from the inequality $|T(p)| \geqslant \gamma$ that $E(f) \geqslant \gamma$.

In the region where at least one of the $|T(p_j)|$ satisfies the inequality $|T(p_j)| \leqslant \gamma$, we make use of the trivial bound

$$|\langle A_{f_1}(t_1) \dots A_{f_s}(t_s)\rangle_\Gamma - \langle A_{f_1}(t_1) \dots A_{f_s}(t_s)\rangle_{\Gamma_s}| \leqslant 2.$$

Thus, we can always write

$$|\mathcal{D}_{p_1\sigma_1; \dots ; p_s\sigma_s}| \leqslant \xi_V^{(s)} + 2^{s+1}\frac{Q}{\gamma}\sqrt{\varepsilon_V + \frac{M_2}{4V}}\prod_{j=1}^{s} \bar{\theta}_\gamma\{T(p_j)\}$$
$$+2\sum_{j=1}^{s}\{1-\bar{\theta}_\gamma(T(p_j))\},$$

where

$$\xi_V^{(s)} = 2^{(s+1)/2}\left\{\frac{M_1M_2}{4V} + M_1\left(\sqrt{\varepsilon_V} + \frac{s-1}{V}\sqrt{M_1}\right)^2\right\}^{1/2}\sum_{j=1}^{s-1} j\,|t_j - t_{j+1}|,$$

$$\bar{\theta}_\gamma(x) = 1, \quad \text{if} \quad |x| \geqslant \gamma,$$
$$\bar{\theta}_\gamma(x) = 0, \quad \text{if} \quad |x| < \gamma.$$

Hence it follows that

$$|\mathcal{D}_{p_1\sigma_1; \dots ; p_s\sigma_s}| \leqslant \xi_V^{(s)} + 2^{s+1}\frac{Q}{\gamma}\sqrt{\varepsilon_V + \frac{M_2}{4V}}$$
$$+2\sum_{j=1}^{s}\{1-\bar{\theta}_\gamma(T(p_j))\}.$$

Starting from inequality (2.21) and repeating, with this modified bound, all the previous arguments, we obtain the following inequality in place of inequality (2.24):

$$|\mathcal{D}_V(h)| \leqslant C(s, \mathcal{K}, M_h)\left\{\left(\frac{M_1M_2}{4V} + M_1\left[\sqrt{\varepsilon_V} + \frac{s-1}{V}\sqrt{M_1}\right]^2\right)^{1/2}\right.$$
$$\left.\times\sum_{j=1}^{s-1} j\ |t_j - t_{j+1}| + 2^{(s+1)/2}\,\frac{Q}{\gamma}\sqrt{\varepsilon_V + \frac{M_2}{4V}}\right\} + \Delta, \tag{2.26}$$

where

$$\Delta = \mathcal{K}_h l(s) 4\sum_{j=1}^{s/2}\left\{\frac{1}{V^{s/2}}\sum_{q_1, \dots, q_{s/2}}\{1-\bar{\theta}_\gamma(T(q_j))\}\prod_{k=1}^{s/2}\frac{1}{(q_k^2+M_h)^2}\right\}.$$

But, obviously,

$$
\begin{aligned}
&\frac{1}{V^{s/2}}\sum_{q_1,\ldots,q_{s/2}}\{1-\bar{\theta}_\gamma(T(q_j))\}\prod_{k=1}^{s/2}\frac{1}{(q_k^2+M_h)^2}\\
&\quad=\frac{1}{V^{s/2}}\sum_{q_1,\ \ldots,\ q_{s/2}}\{1-\bar{\theta}_\gamma(T(q_1))\}\prod_{k=1}^{s/2}\frac{1}{(q_k^2+M_h)^2}\\
&\quad=\frac{1}{V}\sum_{q}\{1-\bar{\theta}_\gamma(T(q))\}\frac{1}{(q^2+M_h)^2}\left\{\frac{1}{V}\sum_{q}\frac{1}{(q^2+M_h)^2}\right\}^{(s/2)-1}\\
&\quad\leqslant\{C(M_h)\}^{(s/2)-1}\frac{1}{V}\sum_{\substack{q\\(|(q^2/2m)-\mu|<\gamma)}}\frac{1}{(q^2+M_h)^2}\,.
\end{aligned}
$$

Further, we have

$$
\frac{1}{V}\sum_{\substack{q\\(|(q^2/2m)-\mu|\leqslant\gamma)}}\frac{1}{(q^2+M_h)^2}\leqslant\frac{1}{(2\pi)^3}\int\limits_{(|(q^2/2m)-\mu|\leqslant\gamma)}\frac{dq}{(q^2+M_h)^2}+\frac{\tilde{C}}{V^{2/3}}
$$

where $\tilde{C}$ is a constant, and

$$
\begin{aligned}
\frac{1}{(2\pi)^3}\int\limits_{(|(q^2/2m)-\mu|\leqslant\gamma)}\frac{dq}{(q^2+M_h)^2}&=\frac{4\pi}{(2\pi)^3}\int\limits_{(|(q^2/2m)-\mu|\leqslant\gamma)}\frac{q^2\,dq}{(q^2+M_h)^2}\\
&=\frac{1}{(2\pi)^2}\int\limits_{(|(q^2/2m)-\mu|\leqslant\gamma)}\frac{|q|\,dq^2}{(q^2+M_h)^2}<\frac{4m\gamma}{(2\pi)^2M_h^{3/2}}\,.
\end{aligned}
$$

Thus, from (2.26) we obtain

$$
\begin{aligned}
|\mathcal{D}_V(h)|\leqslant C(s,\mathcal{K}_h,M_h)&\left\{\left(\frac{M_1M_2}{4V}+M_1\left[\sqrt{\varepsilon_V}+\frac{s-1}{V}\sqrt{M_1}\right]^2\right)^{1/2}\right.\\
&\left.\times\sum_{j=1}^{s-1}j\,|t_j-t_{j+1}|+2^{(s+1)/2}\frac{Q}{\gamma}\sqrt{\varepsilon_V+\frac{M_2}{4V}}\right\}\qquad(2.27)\\
&+2\mathcal{K}_hl(s)\,s[C(M_h)]^{(s/2)-1}\left(\frac{m\gamma}{\pi^2M_h^{3/2}}+\frac{\tilde{C}}{V^{2/3}}\right).
\end{aligned}
$$

The quantity γ can be chosen arbitrarily. Here, we shall put

$$\gamma = k\left(\varepsilon_V + \frac{M_2}{4V}\right)^{1/4},$$

where k is some constant. Then we shall find from (2.27) that if $\varepsilon_V \to 0$ ($V \to \infty$) uniformly in the temperature interval ($0 < \theta \leqslant \theta_0$), then the right-hand side of the inequality (2.27) will also tend to zero uniformly in this interval as $V \to \infty$. In the same way, analogous observations can also be made in later arguments of this chapter.

§ 6. GENERALIZED ASYMPTOTIC RELATIONS FOR THE GREEN'S FUNCTIONS

We turn now to the Green's functions. First of all, we shall consider a so-called T-product of our field functions, which we shall define here as follows:

$$\begin{aligned} &T\{\varphi_{\sigma_1}(\mathbf{r}_1, t_1) \dots \varphi_{\sigma_s}(\mathbf{r}_s, t_s)\\ &\quad = \sum_{(\mathcal{P})} \eta_{\mathcal{P}} \mathcal{P}\{\theta(t_1-t_2) \dots \theta(t_{s-1}-t_s)\varphi_{\sigma_1}(\mathbf{r}_1, t_1) \dots \varphi_{\sigma_s}(\mathbf{r}_s, t_s)\} \end{aligned} \tag{2.28}$$

where

$$\theta(t) = \begin{cases} 1, & t > 0 \\ 0, & t \leqslant 0. \end{cases} \tag{2.29}$$

$\mathcal{P}$ denotes a permutation of the field functions in the field-function product, accompanied by the corresponding permutation of t_1, t_2, ..., t_s in the product of θ-functions; e.g.

$$\begin{aligned} &\mathcal{P}\{\theta(t_1-t_2) \dots \theta(t_{s-1}-t_s)\, \varphi_{\sigma_1}(\mathbf{r}_1, t_1) \dots \varphi_{\sigma_s}(\mathbf{r}_s, t_s)\}\\ &\quad = \eta_p\, \theta(t_{j_1}-t_{j_2}) \dots \theta(t_{j_{s-1}}-t_{j_s})\, \varphi_{\sigma_{j_1}}(\mathbf{r}_{j_1}, t_{j_1}) \dots \varphi_{\sigma_{j_s}}(\mathbf{r}_{j_s}, t_{j_s}). \end{aligned}$$ [†]

$\eta_{\mathcal{P}} = \pm 1$, according to whether the permutation $\mathcal{P}$ is even or odd. The summation runs over all permutations $\mathcal{P}$.

[†] Here the indices $j_1, j_2, \dots, j_s$ are obtained from the set of indices 1, 2, ..., s by means of the permutation $\mathcal{P}$.

Arranging $t_1, t_2, \dots, t_s$ in decreasing order from left to right

$$t_{j_1} > t_{j_2} > \dots > t_{j_s},$$

we have

$$T\{\varphi_{\sigma_1}(\boldsymbol{r}_1, t_1) \dots \varphi_{\sigma_s}(\boldsymbol{r}_s, t_s)\} = \pm\varphi_{\sigma_{j_1}}(\boldsymbol{r}_{j_1}, t_{j_1}) \dots \varphi_{\sigma_{j_s}}(\boldsymbol{r}_{j_s}, t_{j_s}). \tag{2.30}$$

It is clear from (2.28) and (2.29) that the T-product introduced here will vanish if any two time arguments t_i, t'_i coincide. Since we can always substitute the product (2.30) into inequality (2.24), we see that

$$\begin{aligned}&\int h(\boldsymbol{r}_1, \dots, \boldsymbol{r}_s)\left\{\langle T\{\varphi_{\sigma_1}(\boldsymbol{r}_1, t_1) \dots \varphi_{\sigma_s}(\boldsymbol{r}_s, t_s)\}\rangle_{\Gamma}\right.\\ &\quad\left.-\langle T\{\varphi_{\sigma_1}(\boldsymbol{r}_1, t_1) \dots \varphi_{\sigma_s}(\boldsymbol{r}_s, t_s)\}\rangle_{\Gamma_a}\right\} d\boldsymbol{r}_1 \dots d\boldsymbol{r}_s\\ &\leqslant C(s, \mathcal{K}_h, M_h)\left(\left\{\frac{M_1M_2}{4V}+M_1\left[\sqrt{\varepsilon_V}+\frac{s-1}{V}\sqrt{M_1}\right]^2\right\}^{1/2}\right.\\ &\quad\left.\times\sum_{j=1}^{s-1} j\,|t_j-t_{j+1}|+2^{(s+1)/2}\mathcal{K}\sqrt{\varepsilon_V+\frac{M_2}{4V}}\right),\end{aligned} \tag{2.31}$$

and, consequently, for the many-time Green's functions

$$\langle T\{\dots\}\rangle,$$

we have generalized asymptotic relations of the form

$$\begin{aligned}&\langle T\{\varphi_{\sigma_1}(\boldsymbol{r}_1, t_1) \dots \varphi_{\sigma_s}(\boldsymbol{r}_s, t_s)\}\rangle_{\Gamma}\\ &\quad-\langle T\{\varphi_{\sigma_1}(\boldsymbol{r}_1, t_1) \dots \varphi_{\sigma_s}(\boldsymbol{r}_s, t_s)\}\rangle_{\Gamma_a} \to 0\\ &\qquad\text{as}\quad V \to \infty.\end{aligned} \tag{2.32}$$

Finally, we take the two-time Green's functions

$$\begin{aligned}&G_{\Gamma, \Gamma_a}(t, \tau \,|\, \boldsymbol{r}_1, \dots, \boldsymbol{r}_k; \boldsymbol{r}_{k+1}, \dots, \boldsymbol{r}_s)\\ &\quad= \theta(t-\tau)\langle[\mathcal{A}(t), \mathcal{B}(\tau)]_{\pm}\rangle_{\Gamma, \Gamma_a},\end{aligned} \tag{2.33}$$

where

$$\begin{aligned}\mathcal{A}(t) &= \varphi_{\sigma_1}(\boldsymbol{r}_1, t) \dots \varphi_{\sigma_k}(\boldsymbol{r}_k, t),\\ \mathcal{B}(\tau) &= \varphi_{\sigma_{k+1}}(\boldsymbol{r}_{k+1}, \tau) \dots \varphi_{\sigma_s}(\boldsymbol{r}_s, \tau).\end{aligned} \tag{2.34}$$

It is clear that we can substitute the field-function products

$$\mathcal{A}(t)\mathcal{B}(\tau), \quad \mathcal{B}(\tau)\mathcal{A}(t)$$

into the relations (2.24) and multiply the result by $\theta(t-\tau)$. We thus obtain the inequality:

$$\left|\int h(\mathbf{r}_1, \ldots, \mathbf{r}_s)\{G_\Gamma - G_{\Gamma_a}\}\, d\mathbf{r}_1 \ldots d\mathbf{r}_s\right|$$
$$\leqslant C(s, \mathcal{K}_h, M_h)\left\{\left(\frac{M_1M_2}{4V} + M_1\left[\sqrt{\varepsilon_V} + \frac{s-1}{V}\sqrt{M_1}\right]^2\right)^{1/2} s\,|t-\tau|\,\theta(t-\tau)\right.$$
$$\left. + 2\theta(t-\tau)2^{(s+1)/2}\mathcal{K}\sqrt{\varepsilon_V + \frac{M_2}{4V}}\right\}, \tag{2.35}$$

whence follows the generalized asymptotic relation:

$$G_\Gamma - G_{\Gamma_a} \to 0 \quad \text{as} \quad V \to \infty. \tag{2.36}$$

To summarize, we can see that the following theorem is valid:

THEOREM III. *If the conditions I are fulfilled, then on the class* $\mathcal{L}$ *of functions* $h(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_s)$ *we have the following generalized asymptotic relations for the correlation functions and Green's functions:*

$$\langle\varphi_{\sigma_1}(\mathbf{r}_1, t_1) \ldots \varphi_{\sigma_s}(\mathbf{r}_s, t_s)\rangle_\Gamma - \langle\varphi_{\sigma_1}(\mathbf{r}_1, t_1) \ldots \varphi_{\sigma_s}(\mathbf{r}_s, t_s)\rangle_{\Gamma_a} \to 0$$
$$\text{as} \quad V \to \infty,$$
$$\langle T\{\varphi_{\sigma_1}(\mathbf{r}_1, t_1) \ldots \varphi_{\sigma_s}(\mathbf{r}_s, t_s)\}\rangle_\Gamma - \langle T\{\varphi_{\sigma_1}(\mathbf{r}_1, t_1) \ldots \varphi_{\sigma_s}(\mathbf{r}_s, t_s)\}\rangle_{\Gamma_a} \to 0$$
$$\text{as} \quad V \to \infty,$$

and also

$$\theta(t-\tau)\langle[\varphi_{\sigma_1}(\mathbf{r}_1, t) \ldots \varphi_{\sigma_k}(\mathbf{r}_k, t), \varphi_{k+1}(\mathbf{r}_{k+1}, \tau) \ldots \varphi_{\sigma_s}(\mathbf{r}_s, \tau)]_\pm\rangle_\Gamma$$
$$-\theta(t-\tau)\langle[\varphi_{\sigma_1}(\mathbf{r}_1, t) \ldots \varphi_{\sigma_k}(\mathbf{r}_k, t), \varphi_{\sigma_{k+1}}(\mathbf{r}_{k+1}, \tau) \ldots \varphi_{\sigma_s}(\mathbf{r}_s, \tau)]_\pm\rangle_{\Gamma_a} \to 0$$
$$\text{as} \quad V \to \infty.$$

It clearly follows from this that to prove the existence of the generalized limits

$$\lim_{V\to\infty}\langle_\Gamma \ldots\rangle$$

of the averages considered here, it is sufficient to prove the existence of the generalized limits

$$\lim_{(V \to \infty)} \langle \ldots \rangle_{\Gamma_a}$$

of the corresponding averages for the trial Hamiltonian Γ_a. We shall now investigate a proof of the existence of these limits.

§ 7. THE EXISTENCE OF GENERALIZED LIMITS

Apart from the conditions I, we shall now have to impose the following additional conditions, which we shall call conditions I′:

1. The functions

   $$\Lambda(f) = \Lambda(\boldsymbol{p}, \sigma) \quad (\sigma = \pm\tfrac{1}{2})$$

   are defined and bounded in the whole space E of points $\boldsymbol{p}$ and are independent of V.
2. The discontinuities of these functions form a set of measure zero in the space E.[†]

These conditions I′ mean that as a result of the passage to the limit ($V \to \infty$), we go over from sums to Riemann integrals. In fact, the integrability in the Riemann sense of some bounded function is ensured by the fact that the set of its discontinuities is of measure zero.

We shall take some function $h(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_2)$ from the class $\mathscr{L}$ and write, using (2.13)

$$\begin{aligned} &\int h(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_s) \langle \varphi_{\sigma_1}(\boldsymbol{r}_1, t_1) \ldots \varphi_{\sigma_s}(\boldsymbol{r}_s, t_s) \rangle_{\Gamma_a} \, d\boldsymbol{r}_1 \ldots d\boldsymbol{r}_s \\ &\quad = \frac{1}{V^{s/2}} \sum_{p_1, \ldots, p_s} \tilde{h}(\varepsilon_1 \boldsymbol{p}_1, \ldots, \varepsilon_s \boldsymbol{p}_s) \langle A_{p_1\sigma_1}(t_1) \ldots A_{p_s\sigma_s}(t) \rangle_{\Gamma_a}. \end{aligned} \tag{2.37}$$

By virtue of the rules of Bloch and de Dominicis, we have

$$\langle A_{p_1\sigma_1}(t_1) \ldots A_{p_s\sigma_s}(t_s) \rangle_{\Gamma a} = \sum \eta \prod \langle A_{p_j\sigma_j}(t_j) \, A_{p_i\sigma_i}(t_i) \rangle_{\Gamma_a}, \tag{2.38}$$

[†] See Shilov and Gurevich (1964).

where $\eta = \pm 1$. The sum here extends over all possible ways of pairing and, therefore, the number of terms in it is bounded by the finite number $l(s)$.

Further, $\prod$ is a product of $s/2$ factors. For a given pairing, $\boldsymbol{p}_i = \boldsymbol{p}_j$ if $\sigma_i = \sigma_j$, and $\boldsymbol{p}_i = -\boldsymbol{p}_j$ if $\sigma_i = -\sigma_j$. We denote the set of left indices $\boldsymbol{p}_j$ in a given product by $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_{s/2}$ respectively. Then, for a given way of pairing, each of the indices $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_s$ will be equal to $\boldsymbol{q}$ or $-\boldsymbol{q}$. For a given way of pairing, these signs are completely determined by the signs of $\sigma_1, \ldots, \sigma_s$. Therefore, for such a pairing, we can write:

$$\eta \tilde{h}(\varepsilon_1 \boldsymbol{p}_1, \ldots, \varepsilon_s \boldsymbol{p}_s) = H(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_{s/2}),$$

where $H(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_{s/2})$ is a continuous function in the space $E^{(s/2)}$, satisfying the inequality

$$|H(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_{s/2})| \leq \mathcal{K}_h \prod_{j=1}^{s/2} \frac{1}{(q_j^2 + M_h)^2}.$$

On the other hand, because of the relations (2.7), which give expressions for binary correlation averages of the form

$$\langle A_{p_j \sigma_j}(t_j)\, A_{p_i \sigma_i}(t_i) \rangle_{\Gamma_a},$$

the function

$$Q(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_{s/2}) = \prod \langle A_{p_j \sigma_j}(t_j)\, A_{p_i \sigma_i}(t_i) \rangle_{\Gamma_a},$$

will be a function which is defined in $E^{(s/2)}$ and independent of V.

We have now from (2.37)

$$\begin{aligned} &\int h(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_s) \langle \varphi_{\sigma_1}(\boldsymbol{r}_1, t_1) \ldots \varphi_{\sigma_s}(\boldsymbol{r}_s, t_s) \rangle_{\Gamma_a} d\boldsymbol{r}_1 \ldots d\boldsymbol{r}_s \\ &= \sum \left\{ \frac{1}{V^{s/2}} \sum_{\boldsymbol{q}_1, \ldots, \boldsymbol{q}_{s/2}} H(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_{s/2}) Q(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_{s/2}) \right\}, \end{aligned} \tag{2.39}$$

where in the first sum $\sum$ the number of terms is bounded by the number $l(s)$, while the second sum

$$\sum_{\boldsymbol{q}_1, \ldots, \boldsymbol{q}_{s/2}} (\ldots)$$

is taken over the quasi-discrete set $E_V^{(s/2)}$.

We note that, since the set of discontinuities of the function $\Lambda(\mathbf{p}, \sigma)$ has zero measure in E, the functions (2.7) will also possess this property. Thus, the discontinuities of the functions

$$H(\mathbf{q}_1, \ldots, \mathbf{q}_{s/2})\, Q(\mathbf{q}_1, \ldots, \mathbf{q}_{s/2})$$

form a set of measure zero in the space $E^{(s/2)}$.

On the other hand, since

$$|Q(\mathbf{q}_1, \ldots, \mathbf{q}_{s/2})| \leqslant 1,$$

we have the bound

$$|H(\mathbf{q}_1, \ldots, \mathbf{q}_{s/2})\, Q(\mathbf{q}_1, \ldots, \mathbf{q}_{s/2})| \leqslant \mathcal{K}_h \prod_{j=1}^{s/2} \frac{1}{(q_j^2+M_h)^2}\,.$$

In view of the above,

$$\frac{1}{V^{s/2}} \sum_{\mathbf{q}_1, \ldots, \mathbf{q}_{s/2}} H(\mathbf{q}_1, \ldots, \mathbf{q}_{s/2}) Q(\mathbf{q}_1, \ldots, \mathbf{q}_{s/2})$$
$$\to \frac{1}{(2\pi)^{3s/2}} \int \cdots \int H(\mathbf{q}_1, \ldots, \mathbf{q}_{s/2})\, Q(\mathbf{q}_1, \ldots, \mathbf{q}_{s/2})\, d\mathbf{q}_1 \ldots d\mathbf{q}_{s/2}$$
$$\text{as} \quad V \to \infty,$$

and therefore, using (2.39), we obtain

$$\begin{gathered}\int h(\mathbf{r}_1, \ldots, \mathbf{r}_s) \langle \varphi_{\sigma_1}(\mathbf{r}_1, t_1) \ldots \varphi_{\sigma_s}(\mathbf{r}_s, t_s) \rangle_{\Gamma_a}\, d\mathbf{r}_1 \ldots d\mathbf{r}_s \\ \to \sum \frac{1}{(2\pi)^{3s/2}} \int H(\mathbf{q}_1, \ldots, \mathbf{q}_{s/2})\, Q(\mathbf{q}_1, \ldots, \mathbf{q}_{s/2})\, d\mathbf{q}_1 \ldots d\mathbf{q}_{s/2} \\ \text{as} \quad V \to \infty,\end{gathered} \tag{2.40}$$

where in the sum $\sum$ there are no more than $l(s)$ terms ($l(s)$ is the number of different ways of pairing). We emphasize here that the order of the field operators in the averages $\langle \ldots \rangle_{\Gamma_a}$ can be chosen arbitrarily.

In addition, the relation (2.40) also holds for any real parameters $t_1, \ldots, t_s$, so that, for example, this relation can be multiplied by products of functions of the form $\theta(t_j - t_k)$. Hence follows the existence, on the class $\mathcal{L}$ of functions h, of the generalized limits

$$\lim_{V\to\infty} \langle \varphi_{\sigma_1}(\boldsymbol{r}_1, t_1) \ldots \varphi_{\sigma_s}(\boldsymbol{r}_s, t_s)\rangle_{\Gamma_a},$$

$$\lim_{V\to\infty} \langle T\{\varphi_{\sigma_1}(\boldsymbol{r}_1, t_1) \ldots \varphi_{\sigma_s}(\boldsymbol{r}_s, t_s)\}\rangle_{\Gamma_a},$$

$$\lim_{V\to\infty} \theta(t-\tau)\langle[\varphi_{\sigma_1}(\boldsymbol{r}_1, t) \ldots \varphi_{\sigma_k}(\boldsymbol{r}_k, t), \varphi_{\sigma_{k+1}}(\boldsymbol{r}_{k+1}, \tau) \ldots \varphi_{\sigma_s}(\boldsymbol{r}_s, \tau)]_{\pm}\rangle_{\Gamma_a}.$$

Thus, we can see that the following theorem is valid:

THEOREM IV. *If the conditions I (Chapter* 1, § 1*) and I′ (Chapter* 2, § 7*) are fulfilled, then, on the class* $\mathcal{L}$ *of functions* $h(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_s)$, *the following generalized limits for the correlation functions and Green's functions exist:*

$$\lim_{V\to\infty} \langle \varphi_{\sigma_1}(\boldsymbol{r}_1, t_1) \ldots \varphi_{\sigma_s}(\boldsymbol{r}_s, t_s)\rangle_{\Gamma} = \lim_{V\to\infty} \langle \varphi_{\sigma_1}(\boldsymbol{r}_1, t_1) \ldots \varphi_{\sigma_s}(\boldsymbol{r}_s, t_s)\rangle_{\Gamma_a};$$

$$\lim_{V\to\infty} \langle T\{\varphi_{\sigma_1}(\boldsymbol{r}_1, t_1) \ldots \varphi_{\sigma_s}(\boldsymbol{r}_s, t_s)\}\rangle_{\Gamma} = \lim_{V\to\infty} \langle T\{\varphi_{\sigma_1}(\boldsymbol{r}_1, t_1) \ldots \varphi_{\sigma_s}(\boldsymbol{r}_s, t_s)\}\rangle_{\Gamma_a};$$

$$\lim_{V\to\infty} \theta(t-\tau)\langle[\varphi_{\sigma_1}(\boldsymbol{r}_1, t) \ldots \varphi_{\sigma_k}(\boldsymbol{r}_k, t), \varphi_{\sigma_{k+1}}(\boldsymbol{r}_{k+1}, \tau) \ldots \varphi_{\sigma_s}(\boldsymbol{r}_s, \tau)]_{\pm}\rangle_{\Gamma}$$
$$= \lim_{V\to\infty} \theta(t-\tau)\langle[\varphi_{\sigma_1}(\boldsymbol{r}_1, t) \ldots \varphi_{\sigma_k}(\boldsymbol{r}_k, t), \varphi_{\sigma_{k+1}}(\boldsymbol{r}_{k+1}, \tau) \ldots \varphi_{\sigma_s}(\boldsymbol{r}_s, \tau)]_{\pm}\rangle_{\Gamma_a}.$$

As we have seen, expressions of the form

$$\lim_{V\to\infty} \langle \ldots \rangle_{\Gamma_a}$$

can be calculated by applying the rules of Bloch and de Dominicis and replacing the quasi-discrete sums by the corresponding integrals.

CHAPTER 3

CORRELATION FUNCTIONS FOR SYSTEMS WITH FOUR-FERMION NEGATIVE INTERACTION

§ 1. CALCULATION OF THE FREE ENERGY FOR MODEL SYSTEMS WITH ATTRACTION

In this chapter we shall study dynamical systems which correspond to attraction of fermions. We shall begin by calculating the free energy for model Hamiltonians with four-fermion interaction. This problem, as we have shown (Bogolyubov, Jr., 1966b), is of great interest in the study of model problems in the theory of superconductivity and serves as an example of an exact calculation of the free energy for model systems of the BCS type (Wentzel, 1960, and Thirring, 1968).[†] The results and upper bounds obtained here also constitute a proof of the Theorem 3.I formulated in this section.

We shall start from the Hamiltonian

$$H = T-2V \sum_{1\leqslant\alpha\leqslant s} J_\alpha J_\alpha^+. \tag{3.1}$$

If we take the following Fermi-operator expressions for the operators T and J_α:

$$T = \sum_f T(f)\, a_f^+ a_f, \quad J_\alpha = \frac{1}{2V}\sum_f \lambda_\alpha(f)\, a_f^+ a_{-f}^\pm, \tag{3.2}$$

[†] This treatment also solves a number of problems raised by Wentzel (1960) concerning the asymptotically exact calculation of the free energy. and can also be applied to certain quasi-spin Hamiltonians (cf. Bogolyubov, Jr., and Shumovskiĭ (1970, 1971) and Shumovskiĭ (1971).

we obtain the usual BCS Hamiltonian

$$H = \sum_{f} T(f)\, a_f^+ a_f - \frac{1}{2V} \sum_{f,f'} \mathcal{J}(f,f')\, a_f^+ a_{-f}^+ a_{-f'} a_{f'}. \tag{3.3}$$

In fact it is not necessary for our discussion that the operators T and J_α have the explicit form (3.2).

It is sufficient to impose the following general conditions:

$$\begin{aligned} &|J_\alpha| \leqslant M_1, &\quad &|TJ_\alpha - J_\alpha T| \leqslant M_2,\\ &|J_\alpha^+ J_\beta - J_\beta J_\alpha^+| \leqslant \frac{M_3}{V}, &\quad &|J_\alpha J_\beta - J_\beta J_\alpha| \leqslant \frac{M_3}{V}, \end{aligned} \tag{3.4}$$

where M_1, M_2 and M_3 are constants as $V \to \infty$ and the symbol $|\ldots|$ denotes the norm of the indicated operators. We assume also that the free energy per unit volume for the Hamiltonian $H = T$ is bounded by a constant and that the number of terms s in the sum (3.1) is fixed.

We thus start from the Hamiltonian (3.1) with the conditions (3.4). We take the trial Hamiltonian to have the usual form

$$H^0 = T - 2V \sum_{1 \leqslant \alpha \leqslant s} (C_\alpha J_\alpha^+ + C_\alpha^* J_\alpha) + 2V \sum_{1 \leqslant \alpha \leqslant s} |C_\alpha|^2. \tag{3.5}$$

Here the C_α are complex constants determined from the condition that the function

$$f_{H^0}(C) = -\frac{1}{V}\, \theta \ln \operatorname{Tr} e^{-H^0/\theta} \tag{3.6}$$

have its absolute minimum value in the domain of all the complex variables $(C_1, \ldots, C_s)$. We shall denote this complex set of points $(C_1, \ldots, C_s)$ by $\{E^s\}$. By making use of the minimizing values of C, we calculate the free energy per unit volume for the trial Hamiltonian:

$$f_{H^0} = \min_{\{E^s\}} f_{H^0}(C). \tag{3.7}$$

We also take the corresponding free energy for the Hamiltonian (3.1):

$$f_H = -\frac{1}{V}\, \theta \ln \operatorname{Tr} e^{-H/\theta}. \tag{3.8}$$

We shall prove that the difference $f_{H^0}-f_H$ tends to zero as $V \to \infty$. For this it is more convenient to consider first the auxiliary problem with the Hamiltonian

$$\Gamma = H-V \sum_{1\leqslant\alpha\leqslant s} (\nu_\alpha J_\alpha+\nu_\alpha^* J_\alpha^+), \tag{3.9}$$

where $\nu_1, \ldots, \nu_s$ are arbitrary non-zero complex parameters. In this problem, the corresponding trial Hamiltonian has the form

$$\Gamma^0 = H^0-V \sum_{1\leqslant\alpha\leqslant s} (\nu_\alpha J_\alpha+\nu_\alpha^* J_\alpha^+). \tag{3.10}$$

The complex quantities $C = (C_1, \ldots, C_s)$ occurring here are also determined from the condition for the absolute minimum of the function

$$f_{\Gamma^0}(C) = -\frac{\theta}{V} \ln \operatorname{Tr} e^{-\Gamma^0/\theta}. \tag{3.11}$$

We shall obtain an upper bound for the difference $f_{\Gamma^0}-f_\Gamma$, and show this difference to be asymptotically small as $V \to \infty$. Here,

$$f_{\Gamma^0} = \min_{\{E^s\}} f_{\Gamma^0}(C)$$

and f_Γ is the free energy per unit volume for the Hamiltonian Γ. Although this bound for $f_{\Gamma^0}-f_\Gamma$ will be found for $|\nu_\alpha| > 0$, it turns out that it is uniform with respect to $\nu_\alpha \to 0$, so that we can then pass to the limit $\nu_\alpha = 0$ $(1 \leqslant \alpha \leqslant s)$. We thereby obtain a bound for $f_{H^0}-f_H$, proving it to be asymptotically small as $V\to \infty$.

We therefore begin by treating the trial Hamiltonian Γ^0. It is not difficult to show that the problem of the absolute minimum of the function (3.11) has a solution and that this absolute minimum is realized for finite values $C_k = C_k^0$ $(1 \leqslant k \leqslant s)$. This can be seen by using the inequalities

$$\begin{aligned} \sum_\alpha \{|C_\alpha|^2+(|C_\alpha|+2M_1)^2\}-4M_1^2 s+\gamma+f_T &\geqslant f_{\Gamma^0}(C) \\ \geqslant \sum_\alpha \{|C_\alpha|^2+(|C_\alpha|-2M_1)^2\}-4M_1^2 s-\gamma+f_T& \\ \geqslant \sum_\alpha |C_\alpha|^2-4M_1^2 s-\gamma+f_T&, \end{aligned} \tag{3.12}$$

$$C = (C_1, \ldots, C_s), \quad \gamma = 2M_1 \sum_\alpha |\nu_\alpha| \qquad (1 \leqslant \alpha \leqslant s).$$

Thus, the function $f_{\Gamma^0}(C)$ has an absolute minimum at some point $C^0 = (C_1^0, \ldots, C_s^0)$.

Because the function $f_{\Gamma^0}(C)$ is continuously differentiable, at the point $C = C^0$ we have

$$\frac{\partial f_{\Gamma^0}(C)}{\partial C_\alpha} = 0 \qquad (1 \leqslant \alpha \leqslant s),$$

i.e. an equation for C_α:

$$C_\alpha = \langle J_\alpha \rangle_{\Gamma^0} = \frac{\operatorname{Tr} J_\alpha e^{-\Gamma^0/\theta}}{\operatorname{Tr} e^{-\Gamma^0/\theta}}.$$

Taking into account the conditions (3.4) on the operators J_α, we obtain $|C_\alpha| \leqslant M_1 = \text{const.}$

We turn now to the derivation of inequalities limiting the difference in the free energies per unit volume $f_{\Gamma^0} - f_\Gamma$ in terms of averages of

$$\frac{\mathcal{A}}{V} = -2 \sum_{1 \leqslant \alpha \leqslant s} (J_\alpha - C_\alpha)(J_\alpha^+ - C_\alpha^*). \tag{3.13}$$

For this we note that $\Gamma = \Gamma^0 + \mathcal{A}$ and introduce the intermediate Hamiltonian

$$\Gamma^t = \Gamma^0 + t\mathcal{A},$$

which for $t = 0$ coincides with the trial Hamiltonian (3.10) and for $t = 1$ coincides with the original Γ (3.9). The constants $C = (C_1, \ldots, C_s)$ occurring in Γ^t are assumed to be fixed and to be independent of the parameter t.

Let us consider the partition function† and free energy for the intermediate Hamiltonian Γ^t:

$$Q_t = \operatorname{Tr} e^{-\Gamma^t/\theta}; \qquad f_t(C_1, \ldots, C_s) = -\frac{\theta}{V} \ln Q_t; \qquad Q_t = e^{-Vf_t/\theta}. \tag{3.14}$$

Differentiating the equality (3.14) twice with respect to t using operator differentiation, we arrive at the formula

$$-\frac{V}{\theta}\frac{\partial^2 f_t}{\partial t^2} + \frac{V^2}{\theta^2}\left(\frac{\partial f_t}{\partial t}\right)^2 = \frac{1}{\theta^2 Q_t}\int_0^1 \operatorname{Tr}\left\{\mathcal{A} e^{-(\Gamma^t/\theta)\tau} \mathcal{A} e^{-(\Gamma^t/\theta)(1-\tau)}\right\} d\tau.$$

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† See footnote added in proof on p. 136.

Taking into account that

$$\frac{\partial f_t}{\partial t} = \frac{1}{V}\,\frac{\operatorname{Tr}\mathcal{A}e^{-\Gamma^t/\theta}}{\operatorname{Tr}e^{-\Gamma^t/\theta}} = \frac{1}{V}\langle\mathcal{A}\rangle_t,$$

we find

$$-\frac{\partial^2 f_t}{\partial t^2} = \frac{1}{\theta V}\left\{\frac{1}{Q_t}\int\limits_0^1 \operatorname{Tr}\mathcal{A}e^{-(\Gamma^t/\theta)\tau}\mathcal{A}e^{-(\Gamma^t/\theta)(1-\tau)}\,d\tau - \langle\mathcal{A}\rangle^2\right\}$$

$$= \frac{1}{\theta V Q_t}\int\limits_0^1 \operatorname{Tr}\left\{\mathcal{B}e^{-(\Gamma^t/\theta)\tau}\mathcal{B}e^{-(\Gamma^t/\theta)(1-\tau)}\right\}d\tau;$$

$$\mathcal{B} = \mathcal{A} - \langle\mathcal{A}\rangle.$$

Going over to a matrix representation in which the Hamiltonian Γ^t is diagonal, we have

$$-\frac{\partial^2 f_t}{\partial t^2} = \frac{1}{\theta V Q_t}\int\limits_0^1 d\tau \sum_{n,m}\mathcal{B}_{nm}\mathcal{B}_{mn}e^{-\{(E^t_m - E^t_n)/\theta\}\tau - E^t_n/\theta}$$

$$= \frac{1}{\theta V Q_t}\int\limits_0^1 d\tau \sum_{n,m}|\mathcal{B}_{nm}|^2 e^{-\{(E^t_m - E^t_n)/\theta\}\tau - E^t_n/\theta} \geqslant 0.$$

Hence it follows, in particular, that $\partial^2 f_t/\partial t^2 \leqslant 0$, and therefore $\partial f_t/\partial t = \langle\mathcal{A}\rangle_t/V$ decreases with increase of the parameter t. Furthermore, taking into account that f_Γ does not depend on C, we have

$$f_{\Gamma^0}(C) - f_\Gamma = -\int\limits_0^1 \frac{\partial f_t}{\partial t}\,dt = -\int\limits_0^1 \frac{\langle\mathcal{A}\rangle_t}{V}\,dt \geqslant 0.$$

Since this relation is true for all $C = (C_1, \ldots, C_s)$, we also have

$$\min_{\{E^s\}} f_{\Gamma^0}(C) \geqslant f_\Gamma, \quad \text{i.e.} \quad f_{\Gamma^0} \geqslant f_\Gamma.$$

We shall integrate the inequality $\langle \mathcal{A} \rangle_{\Gamma_t} \geqslant \langle \mathcal{A} \rangle_{\Gamma}$ $(0 \leqslant t \leqslant 1)$. Substituting the expression (3.13) for $\mathcal{A}$, we convince ourselves that the inequality

$$f_{\Gamma^0}(C) - f_{\Gamma} \leqslant 2 \sum_{1 \leqslant \alpha \leqslant s} \langle (J_\alpha - C_\alpha)(J_\alpha^+ - C_\alpha^*) \rangle_{\Gamma}$$

is true for any $C = (C_1, \ldots, C_s)$. We put here $C_\alpha = \langle J_\alpha \rangle_{\Gamma}$ $(1 \leqslant \alpha \leqslant s)$ and note that

$$f_{\Gamma^0} = \min_{\{E^0\}} f_{\Gamma^0}(C) \leqslant f_{\Gamma^0}(\langle J \rangle_{\Gamma}).$$

Thus,

$$f_{\Gamma^0} - f_{\Gamma} \leqslant f_{\Gamma^0}(\langle J \rangle_{\Gamma}) - f_{\Gamma} \leqslant 2 \sum_{1 \leqslant \alpha \leqslant s} \left\langle (J_\alpha - \langle J_\alpha \rangle_{\Gamma})(J_\alpha^+ - \langle J_\alpha^+ \rangle_{\Gamma}) \right\rangle_{\Gamma}$$

and, finally,

$$0 \leqslant f_{\Gamma^0} - f_{\Gamma} \leqslant 2 \sum_{1 \leqslant \alpha \leqslant s} \left\langle (J_\alpha - \langle J_\alpha \rangle_{\Gamma})(J_\alpha^+ - \langle J_\alpha^+ \rangle_{\Gamma}) \right\rangle_{\Gamma}, \qquad (3.15)^{\dagger}$$

where, as always,

$$f_{\Gamma^0} = \min_{\{E^0\}} f_{\Gamma^0}(C). \qquad (3.16)$$

Let us recall our main problem. We want to show that the difference $f_{\Gamma^0} - f_{\Gamma}$ is asymptotically small as $V \to \infty$. It follows from (3.15) that we shall have solved our problem if we can demonstrate the asymptotic smallness of the average on the right-hand side of (3.15).

Taking into account the main idea of a paper by the author (Bogolyubov, Jr., 1966a), we express this right-hand side in terms of $\partial^2 f / \partial v_\alpha^* \, \partial v_\alpha$. Differentiating, we have

$$-\frac{1}{\theta} \, \frac{\partial^2 f}{\partial v_\alpha^* \, \partial v_\alpha} = \frac{V}{\theta^2} \int_0^1 \frac{\operatorname{Tr} \left(D^{(\alpha)} e^{-(\tau/\theta)\Gamma} D^{+(\alpha)} e^{-\{(1-\tau)/\theta\}\Gamma} \right) d\tau}{\operatorname{Tr} e^{-\Gamma/\theta}},$$

where

$$D^{(\alpha)} = J_\alpha - \langle J_\alpha \rangle \qquad (1 \leqslant \alpha \leqslant s).$$

[†] For convenience, in the remainder of this section we shall omit the subscript in statistical averages over the Hamiltonian Γ, i.e. we shall write $\langle \ldots \rangle \equiv \langle \ldots \rangle_{\Gamma}$.

Going over to the matrix representation in which Γ_1 is diagonal, we find

$$-\frac{1}{\theta}\,\frac{\partial^2 f}{\partial\nu_\alpha^*\,\partial\nu_\alpha}=\frac{V}{Q\theta^2}\sum_{n,m}\int_0^1 D_{nm}^{(\alpha)}e^{-(\tau/\theta)E_m}D_{mn}^{+(\alpha)}e^{-\{(1-\tau)/\theta\}E_n}\,d\tau$$

$$=\frac{V}{Q\theta^2}\sum_{n,m}\left|D_{nm}^{(\alpha)}\right|^2\int_0^1 e^{-(\tau/\theta)E_m-\{(1-\tau)/\theta\}E_n}\,d\tau$$

$$=\frac{V}{Q\theta}\sum_{n,m}\left|D_{nm}^{(\alpha)}\right|^2\frac{e^{-E_m/\theta}-e^{-E_n/\theta}}{E_n-E_m}\geqslant 0.$$

Using Hölder's inequality, we have the following bound:

$$\frac{V}{Q}\sum_{n,m}\left|D_{nm}^{(\alpha)}\right|^2\left|e^{-E_m/\theta}-e^{-E_n/\theta}\right|$$

$$\leqslant\left(-\frac{\partial^2 f}{\partial\nu_\alpha^*\,\partial\nu_\alpha}\right)^{2/3}\left(\frac{V}{Q}\sum_{n,m}\left|D_{nm}^{(\alpha)}\right|^2\left|E_n-E_m\right|^2\left(e^{-E_m/\theta}+e^{-E_n/\theta}\right)\right)^{1/3}.$$

We carry out the simple transformations:

$$\frac{V}{Q}\sum_{n,m}\left|D_{nm}^{(\alpha)}\right|^2\left|E_n-E_m\right|^2\left(e^{-E_m/\theta}+e^{-E_n/\theta}\right)$$

$$=\frac{V}{Q}\operatorname{Tr}e^{-\Gamma/\theta}\{(\Gamma D^{(\alpha)}-D^{(\alpha)}\Gamma)(D^{+(\alpha)}\Gamma-\Gamma D^{+(\alpha)})+(D^{+(\alpha)}\Gamma-\Gamma D^{+(\alpha)})$$

$$\times(\Gamma D^{(\alpha)}-D^{(\alpha)}\Gamma)$$

$$=V\langle(\Gamma J_\alpha-J_\alpha\Gamma)(\Gamma J_\alpha-J_\alpha\Gamma)^{+}+(\Gamma J_\alpha-J_\alpha\Gamma)^{+}(\Gamma J_\alpha-J_\alpha\Gamma)\rangle$$

$$\leqslant 2V\mathcal{M}^2,$$

where

$$\mathcal{M}=M_2+4M_1M_3s+2M_3\sum_{1\leqslant\alpha\leqslant s}|\nu_\alpha|.$$

Hence we obtain

$$\frac{V}{Q}\sum_{n,m}\left|D_{nm}^{(\alpha)}\right|^2\left|e^{-E_m/\theta}-e^{-E_n/\theta}\right|\leqslant\left(-\frac{\partial^2 f}{\partial\nu_\alpha^*\,\partial\nu_\alpha}\right)^{2/3}(2\mathcal{M}^2V)^{1/3}.$$

Furthermore,

$$\frac{V}{Q}\sum_{n,m}\left|D_{nm}^{(\alpha)}\right|^2 e^{-E_n/\theta} \leqslant \theta\frac{V}{Q}\sum_{n,m}\frac{\left|D_{nm}^{(\alpha)}\right|^2}{(E_n-E_m)}\left(e^{-E_m/\theta}-e^{-E_n/\theta}\right)$$
$$+\frac{V}{Q}\sum_{n,m}\left|D_{nm}^{(\alpha)}\right|^2\left|e^{-E_n/\theta}-e^{-E_m/\theta}\right|,$$

where

$$\frac{V}{Q}\sum_{n,m}\left|D_{nm}^{(\alpha)}\right|^2 e^{-E_n/\theta} = \frac{V}{Q}\operatorname{Tr} D^{(\alpha)}D^{+(\alpha)}e^{-\Gamma/\theta} = V\left\langle D^{(\alpha)}D^{+(\alpha)}\right\rangle$$
$$= V\left\langle (J_\alpha-\langle J_\alpha\rangle)(J_\alpha^+-\langle J_\alpha^+\rangle)\right\rangle.$$

Thus, we finally obtain

$$\left\langle (J_\alpha-\langle J_\alpha\rangle)(J_\alpha^+-\langle J_\alpha^+\rangle)\right\rangle$$
$$\leqslant\left(-\frac{\partial^2 f}{\partial v_\alpha^*\,\partial v_\alpha}\right)\frac{\theta}{V}+\frac{(2\mathcal{J}\mathcal{M}^2)^{1/3}}{V^{2/3}}\left(-\frac{\partial^2 f}{\partial v_\alpha^*\,\partial v_\alpha}\right)^{2/3}.$$

Substituting this inequality into (3.15), we find

$$0\leqslant f_{\Gamma_0}-f_\Gamma\leqslant 2\frac{\theta}{V}\sum_{1\leqslant\alpha\leqslant s}\left(-\frac{\partial^2 f}{\partial v_\alpha^*\,\partial v_\alpha}\right)$$
$$+\frac{2}{V^{2/3}}(2\mathcal{J}\mathcal{M}^2)^{1/3}\sum_{1\leqslant\alpha\leqslant s}\left(-\frac{\partial^2 f}{\partial v_\alpha^*\,\partial v_\alpha}\right)^{2/3}. \tag{3.17}$$

Hence we can see that our problem would be solved if we could show that the second derivatives $|\partial^2 f/\partial v_\alpha^*\,\partial v_\alpha|$ are bounded by a constant as $V\to\infty$. Unfortunately, we are unable to prove such a statement. We must start from the boundedness of the first derivatives $|\partial f/\partial v_\alpha| \leqslant M_1$ $(1\leqslant\alpha\leqslant s)$.

Because of this, we develop a method in which it will not be necessary to use the boundedness of the second derivatives and by means of which we can demonstrate the asymptotic smallness of the difference

$$a=f_{\Gamma_0}-f_\Gamma\,.$$

For the following, it will be more convenient to transform to the polar variables r_α, φ_α:

$$r_\alpha = r_\alpha(\nu_\alpha, \nu_\alpha^*), \quad \varphi_\alpha = \varphi_\alpha(\nu_\alpha, \nu_\alpha^*) \qquad (1 \leqslant \alpha \leqslant s)$$

in inequality (3.17). Accordingly,

$$f(\nu_1, \nu_1^*, \ldots, \nu_s, \nu_s^*) \to f(r_1, \varphi_1, \ldots, r_s, \varphi_s).$$

Then

$$\frac{\partial^2 f}{\partial \nu_\alpha^* \, \partial \nu_\alpha} = \frac{1}{4}\left\{\frac{1}{r_\alpha} \frac{\partial}{\partial r_\alpha}\left(r_\alpha \frac{\partial f}{\partial r_\alpha}\right) + \frac{\partial^2 f}{\partial r_\alpha^2} \frac{1}{r_\alpha^2}\right\}. \tag{3.18}$$

We now make use of the inequality

$$\begin{aligned} |a(r_1, \ldots, r_s; \eta_1, \ldots, \varphi_s)| &\leqslant |a(r_1, \ldots, r_s; \varphi_1, \ldots, \varphi_s) \\ &\quad - a(\xi_1, \ldots, \xi_s; \eta_1, \ldots, \eta_s)| + |a(\xi_1, \ldots, \xi_s; \eta_1, \ldots, \eta_s)| \\ &\leqslant \sum_{1 \leqslant \alpha \leqslant s} \left|\frac{\partial a}{\partial r_\alpha}\right|_{\max} |r_\alpha - \xi_\alpha| + \sum_{1 \leqslant \alpha \leqslant s} \left|\frac{\partial a}{\partial \varphi_\alpha}\right|_{\max} |\varphi_\alpha - \eta_\alpha| \\ &\quad + |a(\xi_1, \ldots, \xi_s; \eta_1, \ldots, \eta_s)|, \end{aligned} \tag{3.19}$$

and take

$$r_\alpha + l \leqslant \xi_\alpha \leqslant r_\alpha + 2l, \qquad \varphi_\alpha \leqslant \eta_\alpha \leqslant \varphi_\alpha + \delta_\alpha, \qquad \delta_\alpha = \frac{l}{r_\alpha}, \qquad (1 \leqslant \alpha \leqslant s),$$

so that

$$a(\xi_1, \ldots, \xi_s; \eta_1, \ldots, \eta_s) = \frac{\int\limits_{r_1+l}^{r_1+2l} \cdots \int\limits_{r_s+l}^{r_s+2l} dr_1 \ldots dr_s \int\limits_{\varphi_1}^{\varphi_1+\delta_1} \cdots \int\limits_{\varphi_s}^{\varphi_s+\delta_s} d\varphi_1 \ldots d\varphi_s a(r_1, \ldots, r_s; \varphi_1, \ldots, \varphi_s) \prod\limits_{1 \leqslant \alpha \leqslant s} r r_\alpha}{(1/2)^3 \prod\limits_{1 \leqslant \alpha \leqslant s} [(r_\alpha + 2l)^2 - (r_\alpha + l)^2]\delta_\alpha} \tag{3.20}$$

We note that

$$\left|\frac{\partial f}{\partial r_\alpha}\right| \leqslant 2M_1, \qquad \left|\frac{\partial f}{\partial \varphi_\alpha}\right| \leqslant 2M_1 r_\alpha$$

and
$$\left|\frac{\partial a}{\partial r_\alpha}\right| \leqslant 4M_1, \qquad \left|\frac{\partial a}{\partial \varphi_\alpha}\right| \leqslant 4M_1 r_\alpha. \tag{3.21}$$

Therefore, the first two terms in inequality (3.19) can be bounded as follows:

$$\begin{aligned}\sum_{1\leqslant\alpha\leqslant s}\left|\frac{\partial a}{\partial r_\alpha}\right|_{\max}|r_\alpha-\xi_\alpha|+\sum_{1\leqslant\alpha\leqslant s}\left|\frac{\partial a}{\partial\varphi_\alpha}\right|_{\max}|\varphi_\alpha-\eta_\alpha|&\\ \leqslant 4M_1s\cdot 2l+4M_1ls=12M_1sl.&\end{aligned}\tag{3.22}$$

Starting from the formulae (3.17) and (3.20), we find a bound for the expression $a(\xi_1,\ldots,\xi_s;\ \eta_1,\ \ldots,\ \eta_s)$. We put (3.18) into the right-hand side of the inequality (3.17). We then multiply (3.17) by the product $r_1r_2\ \ldots\ r_s$ and integrate it over all values of the variables $r_1,\ \ldots,\ r_s,\ \varphi_1,\ \ldots,\ \varphi_s$ within the following limits:

$$r_\alpha+l\leqslant r_\alpha\leqslant r_\alpha+2l,\quad \varphi_\alpha\leqslant\varphi_\alpha\leqslant\varphi_\alpha+\delta_\alpha,\quad \delta_\alpha=\frac{l}{r_\alpha}\quad (1\leqslant\alpha\leqslant s)$$

We then obtain

$$\begin{aligned}0&\leqslant\int\ldots\int a(r_1,\ \ldots,\ r_s;\ \varphi_1,\ \ldots,\ \varphi_s)r_1r_2\ldots r_s\,dr_1\ldots dr_s\,d\varphi_1\ \ldots\ d\varphi_s\\ &\leqslant\frac{\theta}{2V}\int\ldots\int\{F_1r_2r_3\ \ldots\ r_s+F_2r_1r_3\ \ldots\ r_s+\ \ldots\\ &\quad+\ \ldots\ +F_sr_1r_2\ \ldots\ r_{s-1}\}\,dr_1\ \ldots\ dr_s\,d\varphi_1\ \ldots\ d\varphi_s\\ &\quad+\frac{\mathscr{M}^{2/3}}{V^{2/3}}\int\ldots\int\{F_1^{2/3}r_1^{1/3}r_2\ \ldots\ r_s+F_2^{2/3}r_1r_2^{1/3}\ \ldots\ r_s+\ \ldots\\ &\quad+\ \ldots\ +F_s^{2/3}r_1r_2\ \ldots\ r_s^{1/3}\}\,dr_1\ \ldots\ dr_s\,d\varphi_1\ \ldots\ d\varphi_s.\end{aligned}\tag{3.23}$$

Here

$$F_\alpha=\frac{\partial}{\partial r_\alpha}\left(r_\alpha\frac{\partial(-f)}{\partial r_\alpha}\right)+\frac{1}{r_\alpha}\ \frac{\partial}{\partial\varphi_\alpha}\left(\frac{\partial(-f)}{\partial\varphi_\alpha}\right)\geqslant 0\qquad(1\leqslant\alpha\leqslant s).$$

By considering the separate terms of the part of (3.23) containing the factor $\theta/2V$, we see that bounds can be found for these by integrating successively over r_α and φ_α in each of them ($1\leqslant\alpha\leqslant s$) and using the inequalities (3.21) to find bounds for the resulting first derivatives

$\partial f/\partial \varphi_\alpha$ and $\partial f/\partial r_\alpha$; for all the terms of this sum, we then find

$$\frac{\theta}{2V}\sum_{1\leqslant\beta\leqslant s}\frac{2M_1(\delta_\beta+2)}{l\cdot 2^{s-1}\delta_\beta}\prod_{1\leqslant\alpha\leqslant s}\{(r_\alpha+2l)^2-(r_\alpha+l)^2\}\,\delta_\alpha\,. \tag{3.24}$$

By applying Hölder's inequality and using analogous reasoning for all the terms in the sum containing the factor $\mathcal{M}^{2/3}/V^{2/3}$, we obtain

$$\frac{\mathcal{M}^{2/3}}{V^{2/3}2^{1/3}}\sum_{1\leqslant\beta\leqslant s}\frac{((\delta_\beta+2)2M_1)^{2/3}}{2^{s-1}l^{2/3}\delta_\beta^{2/3}}\prod_{1\leqslant\alpha\leqslant s}\{(r_\alpha+2l)^2-(r_\alpha+l)^2\}\,\delta_\alpha\,. \tag{3.25}$$

Using the formulae (3.19)–(3.25), we now obtain a bound for $f_{\Gamma^0}-f_\Gamma = a(r_1,\ \ldots,\ r_s;\ \varphi_1,\ \ldots,\ \varphi_s)$:

$$\begin{aligned}0\leqslant f_{\Gamma^0}-f_\Gamma\leqslant 12M_1ls&+\frac{2\theta M_1}{V}\sum_{1\leqslant\beta\leqslant s}\frac{(\delta_\beta+2)}{l\delta_\beta}\\&+\frac{\mathcal{M}^{2/3}}{V^{2/3}}\,(4M_1)^{2/3}\sum_{1\leqslant\beta\leqslant s}\frac{(\delta_\beta+2)^{2/3}}{\delta_\beta^{2/3}\,l^{2/3}}\,.\end{aligned} \tag{3.26}$$

On the other hand, we note that $\delta_\beta = l/r_\beta$ and choose R in such a way,

$$R\geqslant|\nu_1|,\ \ldots,\ |\nu_s|,$$

that $R\geqslant r_\beta\ (\beta = 1,\ \ldots,\ s)$ and $\delta_\beta\geqslant\delta = l/R$. Then, using the obvious inequality $(l+2R)^{2/3}\leqslant l^{2/3}+(2R)^{2/3}$, we find from (3.26)

$$\begin{aligned}0\leqslant f_{\Gamma^0}-f_\Gamma\leqslant 12M_1ls+\frac{2\theta M_1}{Vl^2}s(l+2R)+\frac{\mathcal{M}^{2/3}}{V^{2/3}}s\frac{(4M_1)^{2/3}}{l^{2/3}}\\+\frac{\mathcal{M}^{2/3}}{V^{2/3}}s\frac{(4M_1)^{2/3}}{l^{2/3}}\,2^{2/3}R^{2/3}\,.\end{aligned} \tag{3.27}$$

We now choose l, which is an arbitrary positive quantity, such that

$$12M_1l = \frac{\mathcal{M}^{2/3}(4M_1)^{2/3}}{V^{2/3}l^{2/3}}\,.$$

Then

$$l = \frac{P}{V^{2/5}};\qquad P = \frac{\mathcal{M}^{2/5}}{2^{2/5}3^{3/5}M_1^{1/5}} = \text{const.}$$

Putting this expression for l into the inequality (3.27), we find

$$0 \leqslant f_{\Gamma^0}-f_\Gamma \leqslant 24M_1 s\frac{P}{V^{2/5}}+\frac{2\theta M_1 s}{V^{3/5}P}+\frac{4\theta M_1 sR}{V^{1/5}P^2}$$
$$+\frac{\mathcal{M}^{2/3}}{V^{2/15}P^{4/3}}s(4M_1)^{2/3}2^{2/3}R^{2/3} \quad \text{for} \quad |\nu_\alpha|<R \quad (1 \leqslant \alpha \leqslant s).$$

Hence it is clear that the difference $f_{\Gamma^0}-f_\Gamma$ vanishes as $V \to \infty$.

We note that in the above bound, we can take the limit $\nu_\alpha = 0$ $(1 \leqslant \alpha \leqslant s)$ and finally prove the statement we made earlier about the asymptotic smallness of the difference $f_{H^0}-f_H$:

$$0 \leqslant f_{H^0}-f_H \leqslant 24M_1 s\frac{\bar{P}}{V^{2/5}}+\frac{2\theta M_1 s}{V^{3/5}\bar{P}}, \tag{3.28}$$

where $\bar{P}$ is a simple combination of the original constants M_1, M_2 and M_3. It is also clear that the above bound is uniform as $\theta \to 0$, and, therefore, the inequality (3.28) is valid for $\theta \geqslant 0$.

We have thus proved the following theorem:

THEOREM 3.I. *Let the Hamiltonian of the system be*

$$H = T-2V\sum_{1\leqslant\alpha\leqslant s} g_\alpha J_\alpha J_\alpha^+ \tag{3.29}$$

and let the operators T and J_α in (3.29) *satisfy the following conditions:*

$$\begin{gathered} T = T^+, \quad |J_\alpha| \leqslant M_1, \\ |TJ_\alpha - J_\alpha T| \leqslant M_2, \quad |J_\alpha J_\beta - J_\beta J_\alpha| \leqslant \frac{M_3}{V}, \\ |J_\alpha^+ J_\beta - J_\beta J_\alpha^+| \leqslant \frac{M_3}{V}; \qquad M_1, M_2, M_3 = \text{const.} \end{gathered} \tag{3.30}$$

In addition, let the free energy per unit volume, calculated for the Hamiltonian T, be bounded by a constant:

$$|f(T)| \leqslant \mathcal{M}_0 = \text{const}, \tag{3.31}$$

We construct the trial Hamiltonian

$$H(C) = T-2V\sum_{\alpha} g_\alpha(C_\alpha J_\alpha^+ + C_\alpha^* J_\alpha - C_\alpha C_\alpha^*) \tag{3.32}$$

where $C = (C_1, \ldots, C_s)$, *and* $C_1, \ldots, C_s$ *are complex numbers. Then the following inequalities are valid:*

$$0 \leqslant \min_{(C)} f\{H(C)\} - f(H) \leqslant \varepsilon\left(\frac{1}{V}\right), \tag{3.33}$$[†]

where $\varepsilon(1/V) \to 0$ (as $V \to \infty$) *uniformly with respect to* θ *in the interval* $(0 < \theta \leqslant \theta_0)$ *where* θ_0 *is an arbitrary fixed temperature.*

§ 2. FURTHER PROPERTIES OF THE EXPRESSIONS FOR THE FREE ENERGY

Having formulated Theorem 3.I, we shall now study the question of the existence of the limit

$$\lim_{V \to \infty} f(H).$$

We shall assume that, in addition to the conditions of Theorem 3.I, the following condition is fulfilled. For any complex $C_1, \ldots, C_s$, the limit

$$\lim_{V \to \infty} f\{H(C)\}$$

exists. We shall denote this limit by

$$f_\infty\{H(C)\}.$$

[†] (1) We shall denote the free energy per unit volume for some Hamiltonian A by $f(A)$, or, if we wish to emphasize its dependence on the volume, by $f_V(A)$.

(2) By $\min_{(C)} f(C)$ we shall always mean the ***absolute minimum*** of the function $f(C)$ in the space of all the points C.

(3) $\varepsilon(1/V)$ is given by (cf. (3.28))

$$\varepsilon\left(\frac{1}{V}\right) = \frac{24M_1 s\bar{P}}{\sqrt{g}\,V^{3/5}} + \frac{2\theta M_1 s}{\sqrt{g}\,V^{3/5}\bar{P}},$$

where $\bar{P}$ is a constant and $g > 0$ is the smallest of the g_α.

We put[†]

$$F_V(C) = f\{H(C)\} - 2\sum_{\alpha} g_\alpha C_\alpha C_\alpha^*$$

and note that

$$\frac{\partial F_V(C)}{\partial C_\alpha} = -2g_\alpha \langle J_\alpha^+ \rangle_{H(C)},$$

$$\frac{\partial F_V(C)}{\partial C_\alpha^*} = -2g_\alpha \langle J_\alpha \rangle_{H(C)}.$$

Therefore, from (3.30) we have

$$\left| \frac{\partial F_V(C)}{\partial C_\alpha} \right| \leqslant 2g_\alpha M_1, \qquad \left| \frac{\partial F_V(C)}{\partial C_\alpha^*} \right| \leqslant 2g M_1,$$

whence

$$|F_V(C') - F_V(C'')| \leqslant 4M_1 \sum_{1 \leqslant \alpha \leqslant s} g_\alpha |C'_\alpha - C''_\alpha|. \tag{3.34}$$

Thus, the set of functions

$$\{F_V(C)\} \quad (V \to \infty)$$

s uniformly continuous.

Since we have the convergence

$$F_V(C) \to F_\infty(C) = f_\infty\{H(C)\} - 2\sum_{\alpha} g_\alpha C_\alpha C_\alpha^* \qquad (V \to \infty)$$

at each point C, we see that this convergence will be uniform on the set $\mathscr{M}(R)$ of points C defined by the inequalities:

$$|C_1| \leqslant R, \ldots, |C_s| \leqslant R,$$

for any *fixed* value of R.

Therefore,

$$\begin{aligned} |f_V\{H(C)\} - f_\infty\{H(C)\}| &= |F_V(C) - F_\infty(C)| \\ &\leqslant \eta_V(R) \to 0 \quad (V \to \infty) \end{aligned} \tag{3.35}$$

[†] We stress that the function $F_V(C)$ need not be treated from the standpoint of the theory of functions of a complex variable. It is not difficult to see that, in essence, the function $F_V(C)$ is a function of real variables and that these can be taken as the real and imaginary parts of the variables C_α.

for

$$C \in \mathscr{M}(R).$$

On the other hand, it follows from (3.34) that

$$|F_V(C_1, C_2, \ldots, C_s) - F_V(0, C_2, \ldots, C_s)| \leqslant 4M_1 g_1 |C_1|.$$

Hence, we have

$$\begin{aligned} f\{H(C_1, C_2, \ldots, C_s)\} - f\{H(0, C_2, \ldots, C_s)\} \\ = F_V(C_1, C_2, \ldots, C_s) - F_V(0, C_2, \ldots, C_s) + 2g_1 |C_1|^2 \quad (3.36) \\ \geqslant -4g_1 M_1 |C_1| + 2g_1 |C_1|^2 . \end{aligned}$$

We denote the lower bound of $f\{H(C)\}$ in the space of the points C by

$$\inf_{(C)} f\{H(C)\}.$$

Obviously,

$$f\{H(0, C_2, \ldots, C_s)\} \geqslant \inf_{(C)} f\{H(C)\}$$

and, therefore, it follows from (3.36) that

$$f\{H(C)\} - \inf_{(C)} f\{H(C)\} \geqslant 2g_1 |C_1| (|C_1| - 2M_1).$$

Replacing C_1 by C_α ($\alpha = 1, 2, \ldots, s$) in the above discussion, we find also

$$f\{H(C) - \inf_{(C)} f\{H(C)\} \geqslant 2g_\alpha |C_\alpha| (|C_\alpha| - 2M_1), \quad \alpha = 1, 2 \ldots s.$$

Hence it is clear that if

$$|C_\alpha| > 2M_1$$

for at least one α, then

$$f\{H(C)\} > \inf_{(C)} f\{H(C)\}.$$

Therefore, the lower bound of $f\{H(C)\}$ on the set $\mathscr{M}(2M_1)$ is equal to the lower bound of this function on the whole space of points C. Since $f\{H(C)\}$ is continuous and the set $\mathscr{M}(2M_1)$ is bounded and closed, this lower bound is attained on $\mathscr{M}(2M_1)$, i.e. an absolute minimum of the function under consideration exists and is realized at certain points:[†]

$$C = C^{(V)} \in \mathscr{M}(2M_1).$$

On the other hand, taking (3.34) into account and passing to the limit $V \to \infty$, we find

$$|F_\infty(C') - F_\infty(C'')| \leq 4M_1 \sum_{1 \leq \alpha \leq s} g_\alpha |C'_\alpha - C''_\alpha|.$$

Hence, repeating exactly the above treatment, we see that the function

$$f_\infty\{H(C)\} = F_\infty(C) + 2\sum_\alpha g_\alpha C_\alpha C^*_\alpha$$

also has an absolute minimum in the space of all the points C, which is realized at certain points

$$C = \tilde{C} \in \mathscr{M}(2M_1).$$

From (3.35), we have now:

$$f_\infty\{H(C^{(V)})\} - f\{H(C^{(V)})\} \leq \eta_V(2M_1),$$
$$f\{H(\tilde{C})\} - f_\infty\{H(\tilde{C})\} \leq \eta_V(2M_1).$$

But, by definition of the absolute minimum,

$$f_\infty\{H(C^{(V)})\} \geq f_\infty\{H(\tilde{C})\},$$
$$f\{H(\tilde{C})\} \geq f\{H(C^{(V)})\}.$$

Consequently,

$$f_\infty\{H(\tilde{C})\} - f\{H(C^{(V)})\} \leq \eta_V(2M_1),$$
$$f\{H(C^{(V)})\} - f_\infty\{H(\tilde{C})\} \leq \eta_V(2M_1),$$

or

$$|f_\infty\{H(\tilde{C})\} - \min_{(C)} f\{H(C)\}| \leq \delta_V, |$$

[†] Generally speaking, the point $C^{(V)}$ of the absolute minimum is not unique.

where

$$\delta_V = \eta_V(2M_1) \to 0 \qquad (V \to \infty).$$

Taking Theorem 3.I into account, we finally obtain

$$-\delta_V \leqslant f_\infty\{H(\bar{C})\} - f(H) \leqslant \varepsilon\left(\frac{1}{V}\right) + \delta_V .$$

Thus, we have now proved the following theorem:

THEOREM 3.II. *If the conditions of Theorem* 3.I *are fulfilled, and if for any complex values of* $C_1, \ldots, C_s$ *the limit*

$$f_\infty\{H(C)\} = \lim_{V\to\infty} f\{H(C)\}$$

exists, then:

1. *This limit function has an absolute minimum in the space of all the points* C, *which is realized at certain points*

$$C = \bar{C} \in \mathcal{M}(2M_1).$$

2. *The inequalities*

$$-\delta_V \leqslant f_\infty\{H(\bar{C})\} - f(H) \leqslant \varepsilon\left(\frac{1}{V}\right) + \delta_V$$

are valid, where

$$\varepsilon\left(\frac{1}{V}\right) \to 0, \quad \delta_V \to 0 \quad \text{as} \quad V \to \infty,$$

$$\delta_V = \max_{(C \in \mathcal{M}(2M_1))} |f\{H(C)\} - f_\infty\{H(C)\}|.$$

§ 3. CONSTRUCTION OF ASYMPTOTIC RELATIONS FOR THE FREE ENERGY

We shall now make a special study of those cases when the operators T and J_α in the Hamiltonian (3.29) have the form (3.2).

As can easily be shown, the conditions of Theorem 3.I will be fulfilled in such cases if

$$\begin{aligned} \frac{1}{V}\sum_p |T(p)\,\lambda_\alpha(p,\sigma)| &\leqslant Q_0, \quad \text{(a)} \\ \frac{1}{V}\sum_p |\lambda_\alpha(p,\sigma)| &\leqslant Q_1, \quad \text{(b)} \\ \frac{1}{V}\sum_p |\lambda_\alpha(p,\sigma)|^2 &\leqslant Q_2. \quad \text{(c)} \end{aligned} \tag{3.37}$$

Here, $\alpha = 1, \ldots, s$; $\sigma = \pm\frac{1}{2}$, Q_0, Q_1, Q_2 = are constants. Then, for example, in the inequalities (3.30) we can put

$$M_1 = Q_1, \quad M_3 = Q_2, \quad M_2 = 2Q_0.$$

Here, let the functions $\lambda_\alpha(p,\sigma)$ satisfy, in addition to the inequalities (3.37), the following conditions:

$$|\lambda_\alpha(p,\sigma)| \leqslant \bar{Q}, \qquad \bar{Q} = \text{const.} \tag{3.38}$$

The set of the discontinuities of the functions $\lambda_\alpha(p,\sigma)$ *is a set of measure zero in the space E.* (3.39)

We shall show that in this situation the conditions of Theorem 3.II are also fulfilled

Before proceeding to this problem, we note that the inequalities (3.37) and (3.38) are not independent.

In fact, (3.37c) follows from the inequalities (3.37b) and (3.38). Also, (3.37b) follows from (3.37a) and (3.38). Thus, all the inequalities imposed here on the λ_α are fulfilled if the inequalities (3.37a) and (3.38) are true.

We note further that (3.37a) and (3.38) hold if the λ_α satisfy the inequalities

$$|\lambda_\alpha(p,\sigma)| \leqslant \frac{\mathcal{K}}{(p^2+a)^3}, \tag{3.40}$$

$$\mathcal{K},\ a = \text{const.}$$

We turn now to the question of the fulfilment of the conditions of Theorem 3.II. Since, in the situation being studied, the conditions of

Theorem 3.I are fulfilled, we need only show that for any fixed complex quantities $C_1, \ldots, C_s$, the limit

$$f_\infty\{H(C)\} = \lim_{(V\to\infty)} f_V\{H(C)\} \tag{3.41}$$

exists. For this, we write the operator form of the trial Hamiltonian as follows:

$$\begin{aligned} H(C) = \sum_f T(f)a_f^+a_f - \tfrac{1}{2}\sum_f\{\Lambda(f)a_f^+a_{-f}^+ + \Lambda^*(f)a_{-f}a_f\} \\ +2V\sum_\alpha g_\alpha C_\alpha^* C_\alpha, \end{aligned} \tag{3.42}$$

where

$$\Lambda(f) = 2\sum_\alpha g_\alpha C_\alpha^*\lambda_\alpha(f). \tag{3.43}$$

Going over to the Fermi amplitudes α_f, α_f^+, which are related to the old a_f, a_f^+ by the transformation (1.18), we diagonalize the form (3.42) and obtain

$$\begin{aligned} H(C) &= \sum_f E(f)\alpha_f^+\alpha_f \\ &+V\left\{2\sum_\alpha g_\alpha C_\alpha C_\alpha^* - \frac{1}{2V}\sum_f [E(f)-T(f)]\right\}, \end{aligned} \tag{3.44}$$

where

$$E(f) = \sqrt{T^2(f)+|\Lambda(f)|^2}.$$

Hence, we find

$$\begin{aligned} f_V\{H(C)\} &= -\frac{\theta}{V}\ln \operatorname{Tr} e^{-H(C)/\theta} \\ &= 2\sum_\alpha g_\alpha C_\alpha C_\alpha^* - \frac{1}{2V}\sum_f\{E(f)-T(f)\} \\ &\quad -\frac{\theta}{V}\sum_f \ln(1+e^{-E(f)/\theta}), \end{aligned}$$

or, separating the indices p and σ in the summation,

$$\begin{aligned} f_V\{H(C)\} &= 2\sum_\alpha g_\alpha C_\alpha C_\alpha^* \\ &\quad -\sum_\sigma \frac{1}{2V}\sum_p\left\{E(p,\sigma)-\left(\frac{p^2}{2m}-\mu\right)\right\} \\ &\quad -\theta\sum_\sigma\frac{1}{V}\sum_p \ln\{1+e^{-E(p,\sigma)/\theta}\}. \end{aligned} \tag{3 45}$$

On the other hand, it is not difficult to see that if we have some bounded function $F(p)$, defined everywhere on the space E, whose discontinuities form a set of measure zero, then

$$\frac{1}{V}\sum_{p\in S_r} F(p) \to \frac{1}{(2\pi)^3}\int\limits_{S_r} F(p)\,dp \tag{3.46}$$

for any sphere S_r with arbitrary fixed radius r. In fact, such a function will be Riemann-integrable in the region S_r. For the summation points

$$p = \left(\frac{2\pi n_1}{L}, \frac{2\pi n_2}{L}, \frac{2\pi n_3}{L}\right)$$

we have

$$\Delta p_x \Delta p_y \Delta p_z = \left(\frac{2\pi}{L}\right)^3 = \frac{(2\pi)^3}{V},$$

so that

$$\frac{(2\pi)^3}{V}\sum_{p\in S_r} F(p)$$

will be the Riemann sum for the integral $\int\limits_{S_r} F(p)\,dp$. We note also that if

$$\frac{1}{V}\sum_{p}|F(p)| \leqslant A = \text{const},$$

then

$$\frac{1}{V}\sum_{p\in S_r}|F(p)| \leqslant A.$$

Hence, passing to the limit $V \to \infty$, we have

$$\frac{1}{(2\pi)^3}\int\limits_{S_r}|F(p)|\,dp \leqslant A.$$

Because of the arbitrariness of the radius r, we see that $F(p)$ is an absolutely integrable function in the whole space E, such that

$$\frac{1}{(2\pi)^3}\int|F(p)|\,dp \leqslant A.$$

For a given function $F(p)$, let the following inequality be valid:

$$\frac{1}{V}\sum_{p\in E-S_r}|F(p)|\leqslant\eta_r,$$

where $(E-S_r)$ denotes the set of the points of E lying outside the sphere S_r. η_r does not depend on V, and

$$\eta_r\to 0,\quad (r\to\infty).$$

Then, obviously

$$\frac{1}{V}\sum_p F(p)\to\frac{1}{(2\pi)^3}\int F(p)\,dp. \tag{3.47}$$

In fact, we shall fix an arbitrarily small number $\varepsilon>0$ and choose $r=r_0$ such that

$$\eta_{r_0}\leqslant\frac{\varepsilon}{4}.$$

In view of (3.46), we can find a number V_0 such that, for $V\gg V_0$, the inequality

$$\left|\frac{1}{V}\sum_{p\in S_{r_0}}F(p)-\frac{1}{(2\pi)^3}\int_{S_{r_0}}F(p)\,dp\right|\leqslant\frac{\varepsilon}{2}$$

holds. We have, therefore

$$\begin{aligned}&\left|\frac{1}{V}\sum_p F(p)-\frac{1}{(2\pi)^3}\int F(p)\,dp\right|\\&\quad\leqslant\left|\frac{1}{V}\sum_{p\in S_{r_0}}F(p)-\frac{1}{(2\pi)^3}\int_{S_{r_0}}F(p)\,dp\right|\\&\quad\quad+\frac{1}{V}\sum_{p\in -S_{r_0}+E}|F(p)|+\frac{1}{(2\pi)^3}\int_{-S_{r_0}+E}|F(p)|\,dp\\&\quad\leqslant\frac{\varepsilon}{2}+\frac{\varepsilon}{4}+\frac{\varepsilon}{4}=\varepsilon.\end{aligned}$$

for any $V\gg V_0$, and this establishes the validity of (3.47).

After these trivial remarks, we turn to the expression (3.45). It is clear from (3.43) and the conditions imposed above on the λ_α that

$$|\Lambda(\boldsymbol{p}, \sigma)| \leqslant \Lambda_0 = \text{const},$$

$$\frac{1}{V}\sum_{p} |\Lambda(\boldsymbol{p}, \sigma)|^2 \leqslant \Lambda_1 = \text{const}.$$

We see also that the discontinuities of the function $\Lambda(p, \sigma)$, and consequently also of the function

$$E(p, \sigma) - \left(\frac{p^2}{2m} - \mu\right) = \sqrt{\left(\frac{p^2}{2m} - \mu\right)^2 + |\Lambda(\boldsymbol{p}, \sigma)|^2} - \left(\frac{p^2}{2m} - \mu\right)$$

form a set of measure zero in the space E. Further, for

$$p^2 \geqslant 4m\mu,$$

we have

$$\frac{p^2}{2m} - \mu \geqslant \frac{p^2}{4m},$$

$$0 \leqslant E(p, \sigma) - \left(\frac{p^2}{2m} - \mu\right) \leqslant \frac{|\Lambda(\boldsymbol{p}, \sigma)|^2}{2\frac{p^2}{4m}} = \frac{2m}{p^2}|\Lambda(\boldsymbol{p}, \sigma)|^2.$$

Hence,

$$\frac{1}{V}\sum_{\substack{p \in E - S_r \\ (p^2 \geqslant 4m\mu)}} \left\{E(p, \sigma) - \left(\frac{p^2}{2m} - \mu\right)\right\} \leqslant \frac{2m}{p^2}\Lambda_1.$$

Thus, taking into account the remarks made above, we see that

$$\frac{1}{V}\sum_{p}\left\{E(p, \sigma) - \left(\frac{p^2}{2m} - \mu\right)\right\}$$
$$\rightarrow \frac{1}{(2\pi)^3}\int\left\{E(p, \sigma) - \left(\frac{p^2}{2m} - \mu\right)\right\} d\boldsymbol{p}.$$

Further, we have

$$\ln\{1 + e^{-\{E(p, \sigma)\}/\theta}\} < e^{-E(p, \sigma)/\theta} \leqslant \text{const}\, e^{-p^2/2m\theta}.$$

Since this function decreases sufficiently rapidly as $p \to \infty$ and its discontinuities form a set of measure zero, we also obtain

$$\frac{1}{V}\sum_p \ln\{1+e^{-E(p,\sigma)/\theta}\} \to \frac{1}{(2\pi)^3}\int \ln\{1+e^{-E(p,\sigma)/\theta}\}\,dp.$$

This also proves the validity of the property (3.41). We have here

$$\begin{aligned} f_\infty\{H(C)\} &= 2\sum_\alpha g_\alpha C_\alpha C_\alpha^* - \frac{1}{2}\sum_\sigma \frac{1}{(2\pi)^3}\int\left\{E(p,\sigma)-\left(\frac{p^2}{2m}-\mu\right)\right\}dp \\ &\quad -\theta\sum_\sigma\frac{1}{(2\pi)^3}\int \ln\{1+e^{-E(p,\sigma)/\theta}\}\,dp, \end{aligned}$$

or, more compactly,

$$\begin{aligned} f_\infty\{H(C)\} &= 2\sum_\alpha g_\alpha C_\alpha^* C_\alpha - \frac{1}{2(2\pi)^3}\int\{E(f)-T(f)\}\,df \\ &\quad -\frac{\theta}{(2\pi)^3}\int \ln\{1+e^{-E(f)/\theta}\}\,df. \end{aligned} \tag{3.48}$$

Here, the integration

$$\int(\ldots)\,df \quad \text{implies the operation} \quad \sum_\sigma\int(\ldots)\,dp.$$

Thus, in the case under investigation, if the conditions (3.37a), (3.38) and (3.39) are fulfilled, the conditions of Theorem 3.II are satisfied. As was noted above, in the proof of this theorem, the convergence

$$f_V\{H(C)\}-f_\infty\{H(C)\} \to 0, \quad (V\to\infty) \tag{3.49}$$

is uniform on any *bounded* set of points C.

§ 4. ON THE UNIFORM CONVERGENCE WITH RESPECT TO θ OF THE FREE ENERGY FUNCTION AND ON BOUNDS FOR THE QUANTITIES δ_V

We shall show that, in the above case, the convergence (3.49) is also uniform with respect to θ in the interval $(0<\theta\leqslant\theta_0)$, where θ_0 is any fixed temperature. It can be seen that this property will be established once we have shown that

$$\left|\frac{\partial}{\partial\theta}\left(f_V\{H(C)\}-f_\infty\{H(C)\}\right)\right| \leqslant X = \text{const} \quad (0<\theta\leqslant\theta_0) \quad (3.50)$$

and this will be our aim now. We have

$$\frac{\partial}{\partial\theta}f_V\{H(C)\} = -\frac{1}{V}\sum_f \ln\{1+e^{-E(f)/\theta}\}-\frac{1}{\theta V}\sum_f\frac{E(f)e^{-E(f)/\theta}}{1+e^{-E(f)/\theta}}.$$

Since

$$\frac{E}{\theta}\,e^{-E/2\theta}\leqslant\frac{2}{e},$$

we can write

$$\left|\frac{\partial}{\partial\theta}f_V\{H(C)\}\right| \leqslant \frac{1}{V}\sum_f e^{-E(f)/\theta}+\frac{2}{e}\,\frac{1}{V}\sum_f e^{-E(f)/2\theta}$$
$$\leqslant\left(1+\frac{2}{e}\right)\frac{1}{V}\sum_f e^{-E(f)/2\theta_0}.$$

Completely analogously, we find

$$\left|\frac{\partial}{\partial\theta}f_\infty\{H(C)\}\right| \leqslant \left(1+\frac{2}{e}\right)\frac{1}{(2\pi)^3}\int e^{-E(f)/2\theta_0}\,df$$

and, therefore,

$$\left|\frac{\partial}{\partial\theta}\left(f_V\{H(C)\}-f_\infty\{H(C)\}\right)\right|$$
$$\leqslant\left(1+\frac{2}{e}\right)\left\{\frac{1}{V}\sum_f e^{-E(f)/2\theta_0}+\frac{1}{(2\pi)^3}\int e^{-E(f)/2\theta_0}\,df\right\}.$$

In view of the rapid falling off of $e^{-E(f)/2\theta_0}$ as $|p|\to\infty$, the integral

$$\int e^{-E(f)/2\theta_0}\,df$$

has a finite value, and

$$\frac{1}{V}\sum_f e^{-E(f)/2\theta_0}\to\frac{1}{(2\pi)^3}\int e^{-E(f)/2\theta_0}\,df.$$

Inequality (3.50) is thus established, and the uniformity of the convergence (3.49) with respect to θ in the interval $(0 < \theta \leqslant \theta_0)$ is thereby also proved. Therefore, in the case under consideration, in Theorem 3.II the relation

$$\delta_V \to 0 \quad (V \to \infty)$$

holds uniformly with respect to θ in this interval.

Thus, putting

$$\varepsilon\left(\frac{1}{V}\right)+\delta_V = \bar{\delta}_V ,$$

we can formulate paragraph 2 of Theorem 3.II in the form

$$|f_\infty\{H(C)\}-f(H)| \leqslant \bar{\delta}_V , \tag{3.51}$$

$\bar{\delta}_V \to 0$ $(V \to \infty)$ uniformly with respect to θ in the interval $(0 < \theta \leqslant \theta_0)$.

An explicit expression for $\varepsilon(1/V)$ has been obtained. (See footnote † on page 101.) It would also not be difficult to obtain an explicit expression for the bound δ_V of the difference

$$f_V\{H(C)\}-f_\infty\{H(C)\} \tag{3.52}$$

if we impose on the $\lambda_\alpha(f)$ the appropriate conditions of smoothness and falling off as $|p| \to \infty$. In fact, as we have seen, (3.52) is the difference between a Riemann sum and the corresponding integral, so that here we can make use of well-known techniques from the theory of the approximate calculation of three-dimensional integrals.

Thus, we can show, for example, that if the functions λ_α of the point (p) are continuous and differentiable everywhere, with the possible exception of certain sufficiently smooth discontinuity surfaces, and go to zero, together with $\partial\lambda_\alpha/\partial p$, sufficiently rapidly as $p \to \infty$, then

$$\delta_V \leqslant \frac{\text{const}}{L} = \frac{\text{const}}{V^{1/3}} .$$

In the case when the λ_α are everywhere continuous, possess derivatives of second order with respect to (p), and, as $p \to \infty$, go to zero along with their derivatives of up to and including second order, we can obtain the stronger bound:

$$\delta_V \leqslant \frac{\text{const}}{V^{2/3}}.$$

§ 5. PROPERTIES OF PARTIAL DERIVATIVES OF THE FREE ENERGY FUNCTION. THEOREM 3.III

We shall now study the partial derivatives of the function

$$f_\infty\{H(C)\} = 2\sum_\alpha g_\alpha C_\alpha^* C_\alpha - \frac{1}{2(2\pi)^3}\int \{E(f)-T(f)+2\theta \ln(1+e^{-E(f)/\theta})\}\, df \tag{3.53}$$

with respect to the variables $C_1, \ldots, C_s, C_1^*, \ldots, C_s^*$. We have

$$U = \frac{\partial}{\partial C_\alpha}\{E(f)-T(f)+2\theta \ln(1+e^{-E(f)/\theta})\}$$
$$= \left\{1-\frac{2}{1+e^{E(f)/\theta}}\right\}\frac{\partial E(f)}{\partial C_\alpha} = 2\,\frac{\tanh\,[E(f)/2\theta]}{E(f)}\left\{\sum_\beta g_\beta C_\beta^* \lambda_\beta(f)\right\} g_\alpha \lambda_\alpha^*(f).$$

But, by virtue of (3.38) and the inequality

$$0 \leqslant \frac{\tanh x}{x} \leqslant 1,$$

we see that U is a bounded function of (p) in E:

$$|U| \leqslant \frac{1}{\theta}\sum_\beta g_\beta |C_\beta|\, \bar{Q}^2 g_\alpha.$$

It is clear also that U is a continuous and differentiable function of C in the whole space of points (C). On the other hand, since $|\tanh x| \leqslant 1$, we also have

$$|U| \leqslant \frac{2}{E(f)}\sum_\beta g_\alpha g_\beta |C_\beta| \cdot |\lambda_\beta(f)\, \lambda_\alpha^*(f)|$$
$$\leqslant \frac{1}{\left|\dfrac{p^2}{2m}-\mu\right|}\sum_\beta g_\alpha g_\beta |C_\beta| \cdot |\lambda_\beta(f)\, \lambda_\alpha^*(f)|.$$

Hence, it is not difficult to see that $U(p)$ is absolutely integrable in E, and

$$\left|\int U\,dp - \int_{S_r} U\,dp\right| = \left|\int_{E-S_r} U\,dp\right| \leqslant \int_{E-S_r} |U|\,dp$$

$$\leqslant \frac{4mg_\alpha}{r^2} \sum_\beta g_\beta |C_\beta| \sqrt{\int |\lambda_\beta(f)|^2\,dp \int |\lambda_\alpha(f)|^2\,dp}$$

for

$$r^2 \geqslant 4m\mu.$$

Consequently,

$$\int_{S_r} U\,dp \to \int U\,dp \quad (r \to \infty)$$

uniformly with respect to C on any bounded set of points C.

Thus, the expression (3.53) can be differentiated with respect to C_α (or C_α^*) under the integral sign, and the corresponding derivatives

$$\frac{\partial f_\infty\{H(C)\}}{\partial C_\alpha} = 2g_\alpha C_\alpha^* - \frac{g_\alpha}{(2\pi)^3} \int \frac{\tanh\,[E(f)/2\theta]}{E(f)} \left\{\sum_\beta g_\beta C_\beta^* \lambda_\beta(f)\right\} \lambda_\alpha^*(f)\,df,$$

$$\frac{\partial f_\infty\{H(C)\}}{\partial C_\alpha^*} = 2g_\alpha C_\alpha - \frac{g_\alpha}{(2\pi)^3} \int \frac{\tanh\,[E(f)/2\theta]}{E(f)} \left\{\sum_\beta g_\beta C_\beta \lambda_\beta^*(f)\right\} \lambda_\alpha(f)\,df \tag{3.54}$$

will be continuous functions of C in the whole space of points (C).

It is not difficult to see that an analogous treatment is valid for partial derivatives of $f_\infty\{H(C)\}$ of any order with respect to the variables $C_1, \ldots, C_s, C_1^*, \ldots, C_s^*$.

In fact, on further differentiation of the expression for U, in addition to the factor $\{\tanh\,[E(f)/2\theta]\}/E(f)$, the expressions

$$\left\{\frac{1}{E}\frac{\partial}{\partial E}\left(\frac{\tanh\,[E/2\theta]}{E}\right)\right\}_{E=E(f)},$$

$$\left\{\frac{1}{E}\frac{\partial}{\partial E}\left(\frac{1}{E}\frac{\partial}{\partial E}\,\frac{\tanh\,[E/2\theta]}{E}\right)\right\}_{E=E(f)}$$

also appear. These are bounded functions of E (since $\{\tanh\,[E/2\theta]\}/E$

can, for small E, be expanded in a Taylor series in *even* powers of E), and, as $E \to \infty$, fall off like

$$\frac{1}{E^2} \sim \frac{\text{const}}{p^4}, \quad \frac{1}{E^3} \sim \frac{\text{const}}{p^6}, \dots .$$

Moreover, on differentiation of U with respect to the variables C, there appear further polynomials in C_α, C_α^* and $\lambda_\alpha(f)$, which also fail to invalidate the above arguments.

Returning to the expressions (3.54) for the first derivatives, we see that, since they are continuous functions of C at the points $C = \bar{C}$ at which the absolute minimum of the function $f_\infty\{H(C)\}$ occurs, we have

$$\begin{aligned} &2g_\alpha C_\alpha^* - \frac{g_\alpha}{(2\pi)^3} \int \frac{\tanh\,[E(f)/2\theta]}{E(f)} \left\{\sum_\beta g_\beta C_\beta^* \lambda_\beta(f)\right\} \lambda_\alpha^*(f)\, df = 0, \\ &2g_\alpha C_\alpha - \frac{g_\alpha}{(2\pi)^3} \int \frac{\tanh\,[E(f)/2\theta]}{E(f)} \left\{\sum_\beta g_\beta C_\beta \lambda_\beta^*(f)\right\} \lambda_\alpha(f)\, df = 0. \end{aligned} \tag{3.55}$$

Thus, summarizing the results just obtained, we see that the following theorem holds:

THEOREM 3.III. *If in the Hamiltonian* (3.29) *the operators* T *and* J_α *have the form* (3.2) *and the functions* $\lambda_\alpha(f)$ *satisfy the conditions* (3.37a), (3.38) *and* (3.39), *then:*

1.
$$|f_V\{H(C)\} - f_\infty\{H(C)\}| \leqslant \delta_V \tag{3.56}$$

for

$$|C_\alpha| \leqslant 2M_1, \quad \alpha = 1, \dots, s$$

where $\delta_V \to 0$ *uniformly with respect to* θ *in the interval* $(0 < \theta \leqslant \theta_0)$.

Here, $f_\infty\{H(C)\}$ is given by the expression (3.53) and possesses continuous partial derivatives of all orders with respect to the variables $C_1, \dots, C_s, C_1^*, \dots, C_s^*$ for all complex values of these variables.

This function has an absolute minimum in the space of all the points (C), which is realized at certain points $C = \bar{C}$:

$$\min_{(C)} f_\infty\{H(C)\} = f_\infty\{H(\bar{C})\},$$

satisfying the equations (3.55).

2. *The inequality*

$$|f_V(H)-f_\infty\{H(\bar{C})\}| \leqslant \bar{\delta}_V \tag{3.57}$$

holds, where $\bar{\delta}_V = (\varepsilon(1/V)+\delta_V) \to 0$ *uniformly with respect to* θ *in the interval* $(0 < \theta \leqslant \theta_0)$.

We shall now add a rider to this theorem.

§ 6. RIDER TO THEOREM 3.III AND CONSTRUCTION OF AN AUXILIARY INEQUALITY

The point $C = \bar{C}$ at which the function $f_\infty\{H(C)\}$ attains an absolute minimum is, in general, not unique. However, in the particular case when the absolute minimum is realized at the point $C = 0$, the uniqueness property holds.

In other words, if

$$\min_{(C)} f_\infty\{H(C)\} = f_\infty\{H(0)\},$$

then

$$f_\infty\{H(C)\} > f_\infty\{H(0)\}$$

for $C \neq 0$ (i.e. for C such that at least one of the components $C_\alpha \neq 0$). To establish this property of the free energy (3.53) taken for the trial Hamiltonian, we shall assume that the opposite is true.

Then there exists a point $\bar{C} \neq 0$ such that

$$f_\infty\{H(\bar{C})\} = f_\infty\{H(0)\}. \tag{3.58}$$

We put

$$C = \sqrt{\tau}\bar{C}, \qquad \tau > 0,$$

and consider the function

$$\phi(\tau) = f_\infty\{H(\sqrt{\tau}\bar{C})\}.$$

Then, because of (3.58),

$$\phi(1) = \phi(0). \tag{3.59}$$

Making use of the expression (3.53) and differentiating, we find:

$$\frac{d\phi(\tau)}{d\tau} = 2\sum_{\alpha} g_\alpha |\bar{C}_\alpha|^2 - \frac{1}{(2\pi)^3}\int \frac{\tanh\,[E(f)/2\theta]}{E(f)} \left|\sum_{\beta} g_\beta C_\beta \lambda_\beta^*(f)\right|^2 df,$$

$$\frac{d^2\phi(\tau)}{d\tau^2} = \frac{4}{(2\pi)^3}\int \frac{e^{E/\theta}}{(1+e^{E/\theta})^2}\,\frac{\sinh\,(E/\theta)-E/\theta}{E^3}\left|\sum_{\beta} g_\beta \bar{C}_\beta \lambda_\beta^*(f)\right|^4 df.$$

Since

$$\frac{\sinh\,(E/\theta)-E/\theta}{E^3} > 0,$$

$d^2\phi/d\tau^2$ can go to zero only if, for all f,

$$\left|\sum_{\alpha} g_\alpha \bar{C}_\alpha \lambda_\alpha^*(f)\right| = 0$$

identically. But in this case,

$$\int \frac{\tanh\,(E/2\theta)}{E}\left|\sum_{\beta} g_\beta \bar{C}_\beta \lambda_\beta^*(f)\right|^2 df = 0$$

also, so that

$$\frac{d\phi(\tau)}{d\tau} = 2\sum_{\alpha} g_\alpha |\bar{C}_\alpha|^2 > 0, \qquad \tau \geqslant 0.$$

But this inequality contradicts (3.59). Consequently,

$$\frac{d^2\phi(\tau)}{d\tau^2} > 0. \tag{3.60}$$

On the other hand, since $C = 0$ gives the absolute minimum of $f_\infty\{H(C)\}$, we have

$$\phi(\tau) \geqslant \varphi(0), \quad \tau > 0.$$

Therefore, $(d\phi(\tau)/d\tau)_{\tau=0}$ cannot be negative:

$$\left(\frac{d\phi(\tau)}{d\tau}\right)_{\tau=0} \geqslant 0.$$

Hence it follows from (3.60) that

$$\frac{d\phi(\tau)}{d\tau} > 0 \quad \text{for} \quad \tau > 0$$

and, consequently,

$$\phi(1) > \phi(0),$$

which again contradicts (3.59). Our rider is thus proved.

To conclude this section, which contains preliminary results relating to the properties of the free energies $f_V(H)$, $f_V\{H(C)\}$ and $f_\infty\{H(C)\}$, we shall prove one more inequality, which we shall use frequently in the following discussions.

We shall consider systems defined by a Hamiltonian which depends linearly on some parameter τ:

$$H_\tau = \Gamma_0 + \tau\Gamma_1 .$$

We shall formally define the expression

$$f_V(H_\tau) = -\frac{\theta}{V} \ln \operatorname{Tr} e^{-H_\tau/\theta},$$

which we shall call the free energy per unit volume for the model system H_τ. Differentiating this expression, we have

$$\frac{d}{d\tau} f_V(H_\tau) = \frac{1}{V} \frac{\operatorname{Tr} \Gamma_1 e^{-H_\tau/\theta}}{\operatorname{Tr} e^{-H_\tau/\theta}} = \frac{1}{V} \langle \Gamma_1 \rangle_{H_\tau} \tag{3.61}$$

and

$$\frac{d^2 f_V(H_\tau)}{d\tau^2} = -\frac{1}{\theta} \int\limits_0^1 \frac{\operatorname{Tr} \{\Gamma_1 e^{-(H_\tau/\theta)\xi} \Gamma_1 e^{-(H_\tau/\theta)(1-\xi)}\}\, d\xi}{\operatorname{Tr} e^{-H_\tau/\theta}},$$

where

$$\bar{\Gamma}_1 = \Gamma_1 - \langle \Gamma_1 \rangle_{H_\tau} .$$

But, as we have shown in § 1 of this chapter,

$$\frac{d^2 f_V(H_\tau)}{d\tau^2} \leqslant 0, \tag{3.62}$$

in view of which,

$$\left\{\frac{df_V(H_\tau)}{d\tau}\right\}_{\tau=1} \leqslant \frac{df_V(H_\tau)}{d\tau} \leqslant \left\{\frac{df_V(H_\tau)}{d\tau}\right\}_{\tau=0} \qquad (0 \leqslant \tau \leqslant 1)$$

and, therefore, for the difference

$$f_V(\Gamma_0+\Gamma_1)-f_V(\Gamma_0) = \int_0^1 \frac{d}{d\tau} f_V(H_\tau)\, d\tau$$

we obtain the inequality

$$\left\{\frac{df_V(H_\tau)}{d\tau}\right\}_{\tau=1} \leqslant f_V(\Gamma_0+\Gamma_1)-f_V(\Gamma_0) \leqslant \left\{\frac{df_V(H_\tau)}{d\tau}\right\}_{\tau=0}.$$

Thus, on the basis of (3.61), we have established the following important inequality:

$$\frac{1}{V}\langle\Gamma_1\rangle_{\Gamma_0+\Gamma_1} \leqslant f_V(\Gamma_0+\Gamma_1)-f_V(\Gamma_0) \leqslant \frac{1}{V}\langle\Gamma_1\rangle_{\Gamma_0}. \qquad (3.63)$$

We shall make use of these inequalities later when we specify concrete model and trial systems and choose the source terms in an appropriate way.

§ 7. ON THE DIFFICULTIES OF INTRODUCING QUASI-AVERAGES

We shall now study the question of the determination of quasi-averages. Let $\mathcal{A}$ be some operator of the type for which the limit Theorems II, III and IV were formulated in Chapters 1 and 2, e.g. a product of Fermi amplitudes, field functions or similar operators.

Then the quasi-average

$$\prec \mathcal{A} \succ_H$$

of such an operator will be defined, for the Hamiltonian (3.29) under consideration, as the limit

$$\prec \mathcal{A} \succ_H = \lim_{\nu\to 0}\left(\lim_{V\to\infty}\langle\mathcal{A}\rangle_\Gamma\right) \qquad (3.64)$$

of an ordinary average

$$\langle \mathcal{A} \rangle_{\Gamma}$$

taken over a Hamiltonian Γ obtained from H by adding "source terms" to it:

$$\begin{aligned}\Gamma &= H - V\sum_{\alpha}(\nu_\alpha J_\alpha^+ + \nu_\alpha^* J_\alpha)\\ &= T - 2V\sum_{\alpha} g_\alpha J_\alpha J_\alpha^+ - V\sum_{\alpha}(\nu_\alpha J_\alpha^+ + \nu_\alpha^* J_\alpha).\end{aligned} \tag{3.65}$$

We now wish to call attention to certain difficulties associated with the definition (3.64). Thus, in the definition given, there is no indication in which region the parameters ν must lie, or how they must tend to zero in order to ensure convergence in the definition (3.64).

We shall show that, even in the simplest cases, if $|\nu|$ tends to zero arbitrarily, the limit $\lim\limits_{\nu \to 0}$ *may not exist.*

We shall take, as an example, the Hamiltonian (1.1):

$$H = T - 2VgJJ^+,$$

which we have examined in a number of papers;[†] the basic results of these were summarized briefly at the beginning of Chapter 1.

We recall that here T and J are given by the formulae (1.2), and the function $\lambda(f)$ satisfies all the conditions imposed in Chapter 1, § 1.

For Γ, we took a Hamiltonian with real positive ν:

$$\Gamma = \Gamma_\nu = T - 2VgJJ^+ - \nu V(J + J^+), \quad (\nu > 0). \tag{3.66}$$

As we have shown, Γ reduces to the form (see (1.11))

$$\begin{aligned}\Gamma &= \Gamma_a - 2Vg(J - C(\nu))(J^+ - C(\nu)),\\ \Gamma_a &= T - \tfrac{1}{2}\sum_f \Lambda(f)\{a_f^+ a_{-f}^+ + a_{-f} a_f\} + 2gVC^2,\\ \Lambda(f) &= 2g\lambda(f)\left\{C(\nu) + \frac{\nu}{2g}\right\}.\end{aligned}$$

[†] Bogolyubov, Jr. (1966 a, c, 1967).

Here,

$$C(\nu)+\frac{\nu}{2g} > 0$$

and the quantity $C = C(\nu)$ realizes the absolute minimum of the function $f_\infty\{\Gamma(C)\}$:

$$\min_{(C)} f_\infty\{\Gamma(C)\} = f_\infty\{\Gamma(C(\nu))\}.$$

Moreover (see (1.13)),

$$\langle (J-C(\nu))(J^+-C(\nu))\rangle_\Gamma \leqslant \varepsilon_V \to 0 \quad (V\to\infty). \tag{3.67}$$

It can be seen that the Hamiltonian Γ belongs to the class (1.14), and, by virtue of the inequality (3.67) and the conditions imposed on $\lambda(f)$, the conditions I of Chapter 1 and I′ of Chapter 2, § 7, are fulfilled.

Because of this, we can make use of the above-mentioned limit theorems and establish the existence of limits of the type

$$\lim_{V\to\infty} \langle \mathcal{A}\rangle_\Gamma = \lim_{V\to\infty} \langle \mathcal{A}\rangle_{\Gamma_a}.$$

We note further that, as has already been pointed out (see (1.7)),

$$C(\nu) \to C(0) = \bar{C} \quad (\nu > 0,\ \nu \to 0). \tag{3.68}$$

On the other hand, the expression

$$\lim_{V\to\infty} \langle \mathcal{A}\rangle_{\Gamma_a}$$

can be expanded in explicit form using the rules of Bloch and de Dominicis, and it can be proved in a completely elementary way that the passage to the limit $\nu \to 0$ $(\nu > 0)$ can be made and reduces simply to replacing $C(\nu)$ by $\bar{C}$ in this expression, i.e. to replacing the averaging over Γ_a by averaging over $H(\bar{C})$.

In this way we can establish the existence of the quasi-averages

$$\prec \mathcal{A} \succ_H = \lim_{\binom{\nu\to 0}{\nu>0}} \lim_{V\to\infty} \langle \mathcal{A}\rangle_\Gamma = \lim_{V\to\infty} \langle \mathcal{A}\rangle_{H(\bar{C})}. \tag{3.69}$$

With the definition (3.64), and for the case when

$$\bar{C} \neq 0, \tag{3.70}$$

we shall now examine how the situation changes when we go over to complex values of ν and, in place of the Hamiltonian (3.66), take

$$\Gamma_{\nu,\nu^*} = T - 2VgJJ^+ - V(\nu J^+ + \nu^* J). \tag{3.71}$$

Here, we shall put $\nu = |\nu| e^{i\varphi}$ and note that Γ_{ν,ν^*} can be reduced to the form $\Gamma = \Gamma_{|\nu|}$ (i.e. to the Hamiltonian (3.66) with $|\nu|$ in place of ν) by means of the gauge transformation

$$a_f \to a_f e^{i(\varphi/2)}, \quad a_f^+ \to a_f^+ e^{-i(\varphi/2)}.$$

Thus, we obtain, for example,

$$\begin{aligned} \langle a_f^+(t)\, a_{-f}^+(\tau)\rangle_{\Gamma_{\nu,\nu^*}} &= e^{-i\varphi} \langle a_f^+(t)\, a_{-f}^+(\tau)\rangle_{\Gamma_{|\nu|}} \\ &= \frac{\nu^*}{|\nu|} \langle a_f^+(t)\, a_{-f}^+(\tau)\rangle_{\Gamma_{|\nu|}} . \end{aligned} \tag{3.72}$$

The limit

$$\lim_{|\nu| \to 0} \lim_{V \to \infty} \langle a_f^+(t)\, a_{-f}^+(\tau)\rangle_{\Gamma_{|\nu|}} \tag{3.73}$$

obviously exists and is given by the formula (1.8) in which $\bar{C}$ replaces C in the expressions for $u(f)$, $v(f)$ and $E(f)$. Then, in the case (3.70) under consideration, the expression (3.73) does not go identically to zero.

Consequently, although

$$\lim_{V \to \infty} \langle a_f^+(t)\, a_{-f}^+(\tau)\rangle_{\Gamma_{\nu,\nu^*}}$$

always exists for $|\nu| > 0$, the limit

$$\lim_{\nu \to 0} \lim_{V \to \infty} \langle a_f^+(t)\, a_{-f}^+(\tau)\rangle_{\Gamma_{\nu,\nu_*}} \tag{3.74}$$

does not, for the trivial reason that the ratio $\nu/|\nu|$ does not tend to any limit as $\nu \to 0$.

The limit (3.74) exists only when ν tends to zero in such a way that the ratio $\nu/|\nu|$ is finite.

In the general case (3.65), with the passage to the limit $\nu \to 0$ the situation is found, naturally, to be even more complicated. Apart from gauge invariance (due to the gauge group), other groups of transformations can also occur, e.g. the rotation group.

We shall now direct our attention to a difficulty which is specific for $s > 1$.

We take the Hamiltonian

$$\begin{aligned} H &= T - 2VgJ_1J_1^\dagger - 2VgJ_2J_2^\dagger, \\ \Gamma &= H - V\{\nu_1(J_1 + J_1^\dagger) + \nu_2(J_2 + J_2^\dagger)\}. \end{aligned}$$

Here, we take ν_1 and ν_2 to be *real* and *positive*.

We put here

$$J_1 = \frac{J}{\sqrt{2}}, \quad J_2 = -\frac{J}{\sqrt{2}},$$

where the operators J and T have the same form as in the Hamiltonians (1.1), and (3.66).

In the given case, H will, in this way, be the same Hamiltonian (1.1) that we have just considered.

We take $\nu_1 = \nu_2$. Then the source terms will drop out completely:

$$\Gamma = H$$

and, since the operator H conserves the number of particles, we have, identically,

$$\langle a_f^\dagger a_{-f}^\dagger \rangle_\Gamma = 0.$$

It can be seen that in such a situation we cannot define a quasi-average correctly at all.

§ 8. A NEW METHOD OF INTRODUCING QUASI-AVERAGES

In order to avoid difficulties of the above type, *we propose that ν be taken proportional to $\bar{C}$ with positive proportionality coefficients:*

$$\nu_\alpha = r_\alpha \bar{C}_\alpha, \quad r_\alpha > 0, \quad \alpha = 1, 2, \ldots, s. \tag{3.75}$$

In such a case, we shall consider the trial Hamiltonian

$$\begin{aligned}\Gamma_a = T-2V\sum_\alpha g_\alpha\{\bar{C}_\alpha J_\alpha^+ + \bar{C}_\alpha^* J_\alpha\}\\ -V\sum_\alpha r_\alpha(\bar{C}_\alpha J_\alpha^+ + \bar{C}_\alpha^* J_\alpha)+\text{const.}\end{aligned} \tag{3.76}$$

Here we shall not write out the constant term, since it affects neither the calculation of the averages $\langle\ldots\rangle_{\Gamma_a}$ nor the equations of motion. It can be seen that we have obtained a trial Hamiltonian for H with the transformed parameters

$$g_\alpha \to g_\alpha+\frac{r_\alpha}{2},$$

$$H_r = T-2V\sum_\alpha\left(g_\alpha+\frac{r_\alpha}{2}\right)J_\alpha J_\alpha^+ .$$

In order that Γ_a from (3.76) be a trial Hamiltonian not for H_r but for the original H, we must replace g_α by $g_\alpha-r_\alpha/2$ in the expression (3.65) for Γ (in which ν is taken in accordance with (3.75)), thereby obtaining

$$\Gamma = T-2V\sum_\alpha\left(g_\alpha-\frac{r_\alpha}{2}\right)J_\alpha J_\alpha^+ - V\sum_\alpha r_\alpha(J_\alpha^+\bar{C}_\alpha+J_\alpha\bar{C}_\alpha^*).$$

It is clear that here, apart from adding "sources", we have performed a "renormalization" of the parameters g_α.

We can add any constant term to this expression for Γ since it will not affect either the average $\langle\ldots\rangle_\Gamma$ or the equations of motion.

As such a constant term, we shall take

$$V\sum_\alpha r_\alpha\bar{C}_\alpha^*\bar{C}_\alpha .$$

Then the Hamiltonian Γ will be represented by the form

$$\begin{aligned}\Gamma = T-2V\sum_\alpha\left(g_\alpha-\frac{r_\alpha}{2}\right)J_\alpha J_\alpha^+ - V\sum_\alpha r_\alpha(J_\alpha^+\bar{C}_\alpha+J_\alpha\bar{C}_\alpha^*)\\ +V\sum_\alpha r_\alpha\bar{C}_\alpha\bar{C}_\alpha^* = H+V\sum_\alpha r_\alpha(J_\alpha-\bar{C}_\alpha)(J_\alpha^+-\bar{C}_\alpha^*).\end{aligned} \tag{3.77}$$

We emphasize that here, as always, $\bar{C}$ denotes the point at which the absolute minimum of the function $f_\infty\{H(C)\}$ (see (3.48)) is attained. For notational convenience, in (3.77) we put

$$r_\alpha = 2\tau_\alpha g_\alpha \quad \text{where} \quad \tau_\alpha > 0.$$

Thus we shall be concerned with a Hamiltonian having the form

$$\begin{gathered} \Gamma = H + 2V\sum_\alpha \tau_\alpha g_\alpha (J - \bar{C}_\alpha)(J_\alpha^+ - \bar{C}_\alpha^*) = T - 2V\sum_\alpha g_\alpha J_\alpha J_\alpha^+ \\ + 2V\sum_\alpha \tau_\alpha g_\alpha (J_\alpha - \bar{C}_\alpha)(J_\alpha^+ - \bar{C}_\alpha^*), \\ \tau_\alpha > 0, \quad \alpha = 1, 2, \ldots, s. \end{gathered} \tag{3.78}$$

We shall show that, with the above choice of Γ, no difficulties will now arise in the definition of the quasi-averages

$$\prec \mathcal{A} \succ_H = \lim_{\tau \to 0} \lim_{V \to \infty} \langle \mathcal{A} \rangle_\Gamma, \quad \tau_\alpha > 0, \quad \alpha = 1, \ldots, s.$$

For this, we note first of all that, for

$$\tau_1 = 1, \ldots, \quad \tau_s = 1,$$

from (3.78) we shall have:

$$\Gamma = T - 2V\sum_\alpha g_\alpha (J_\alpha^+ \bar{C}_\alpha + J_\alpha \bar{C}_\alpha^*) + 2V\sum_\alpha g_\alpha \bar{C}_\alpha \bar{C}_\alpha^* = H(\bar{C})$$

(cf. also formula (3.84)). Since

$$(J_\alpha - \bar{C}_\alpha)(J_\alpha^+ - \bar{C}_\alpha^*) \geqslant 0,$$

we see that

$$\begin{gathered} H(\bar{C}) - \Gamma \geqslant 0, \\ \text{and} \qquad \Gamma - H \geqslant 0 \quad \text{for} \quad 0 < \tau_\alpha < 1. \end{gathered} \tag{3.79}$$

Consequently, the inequalities

$$f_V\{H(C)\} \geqslant f_V(\Gamma) \geqslant f_V(H) \quad (\text{for} \quad 0 < \tau_\alpha < 1) \tag{3.80}$$

are valid. However,

$$0 \leqslant f_V\{H(\bar{C})\} - f_V(H) \leqslant |f_\infty\{H(\bar{C})\} - f_V(H)| + |f_\infty\{H(\bar{C})\} - f_V\{H(\bar{C})\}|.$$

Therefore, on the basis of Theorem 3.III, we obtain

$$0 \leqslant f_V\{H(\bar{C})\} - f_V(H) \leqslant \bar{\delta}_V + \delta_V .$$

Hence, taking (3.80) into account, we find

$$\begin{aligned} 0 &\leqslant f_V(\Gamma) - f_V(H) \leqslant \bar{\delta}_V + \delta_V , \\ 0 &\leqslant f_V\{H(\bar{C})\} - f_V(\Gamma) \leqslant \bar{\delta}_V + \delta_V . \end{aligned} \tag{3.81}$$

We make use now of inequality (3.63) and substitute into it

$$\Gamma_0 = H, \quad \Gamma_1 = \Gamma - H = 2V \sum_{\alpha} \tau_\alpha g_\alpha (J_\alpha - \bar{C}_\alpha)(J_\alpha^+ - \bar{C}_\alpha^*).$$

Then, from the first of the inequalities (3.81), we obtain

$$2 \sum_{\alpha} \tau_\alpha g_\alpha \langle (J_\alpha - \bar{C}_\alpha)(J_\alpha^+ - \bar{C}_\alpha^*) \rangle_\Gamma \leqslant \bar{\delta}_V + \delta_V . \tag{3.82}$$

We have thus proved the following theorem:

THEOREM 3.IV. *Let the conditions of Theorem* 3.III *be fulfilled, and let* Γ *be represented by the expression* (3.78) *in which*

$$0 < \tau_\alpha < 1, \quad \alpha = 1, 2, \ldots, s.$$

Then the following inequalities hold:

$$\begin{aligned} 0 &\leqslant f_V(\Gamma) - f_V(H) \leqslant \bar{\delta}_V + \delta_V \to 0 \quad \text{as} \quad V \to \infty, \\ \sum_{\alpha} g_\alpha \langle (J_\alpha - \bar{C}_\alpha)(J_\alpha^+ - \bar{C}_\alpha^*) \rangle_\Gamma &\leqslant \frac{\bar{\delta}_V + \delta_V}{2\tau_0} \to 0 \quad \text{as} \quad V \to \infty, \end{aligned} \tag{3.83}$$

where τ_0 *is the smallest of the quantities* $\tau_1, \tau_2, \ldots, \tau_s$.

RIDER TO THEOREM 3.IV

We shall consider the more general case when, in the expression

$$H = T - 2V \sum_{\alpha} g_\alpha J_\alpha J_\alpha^+ ,$$

the operators T and J_α are not represented by (3.2) and satisfy only the conditions of Theorem 3.II.

Then, replacing Theorem 3.III by Theorem 3.II in the arguments carried through above, we see that Theorem 3.IV remains true.

We note, further, that in the case when the operators have the specific form (3.2) and the conditions of Theorem 3.III are fulfilled, the Theorem 3.IV proved above enables us to transform the Hamiltonian Γ directly to the form (1.14), (1.15), and the conditions I (Chapter 1, § 1) and I′ (Chapter 2, § 7) are found to be fulfilled.

In fact, we have

$$\begin{aligned} H &= T-2V\sum_\alpha g_\alpha(J_\alpha^+\bar{C}_\alpha+J_\alpha\bar{C}_\alpha^*)+2V\sum_\alpha g_\alpha\bar{C}_\alpha^*\bar{C}_\alpha \\ &\quad -2V\sum_\alpha g_\alpha(J_\alpha-\bar{C}_\alpha)(J_\alpha^+-\bar{C}_\alpha^*), \end{aligned}$$

and, therefore,

$$\Gamma = H(\bar{C})-2V\sum_\alpha g_\alpha(1-\tau_\alpha)(J_\alpha-\bar{C}_\alpha)(J_\alpha-\bar{C}_\alpha^*), \tag{3.84}$$

where

$$H(\overline{C}) = T-2V\sum_\alpha g_\alpha(J_\alpha^+\bar{C}_\alpha+J_\alpha\bar{C}_\alpha^*)+2V\sum_\alpha g_\alpha\bar{C}_\alpha\bar{C}_\alpha^*.$$

It can be seen that this Hamiltonian has the form of the Hamiltonian (1.14), (1.15) in which we put

$$\begin{aligned} \Gamma_a &= H(\overline{C}), \\ \Lambda(f) &= 2\sum_\alpha g_\alpha\lambda_\alpha(f)\bar{C}_\alpha^*, \\ K &= 2V\sum_\alpha g_\alpha\bar{C}_\alpha^*\bar{C}_\alpha, \\ G_\alpha &= 2g_\alpha(1-\tau_\alpha) > 0, \quad C_\alpha = \bar{C}_\alpha. \end{aligned} \tag{3.85}$$

By virtue of Theorem 3.IV, the following inequality is fulfilled:

$$\sum_\alpha G_\alpha\langle(J_\alpha-\bar{C}_\alpha)(J_\alpha-\bar{C}_\alpha^*)\rangle_\Gamma \leqslant \varepsilon_V, \tag{3.86}$$

where

$$\varepsilon_V = \frac{\delta_V+\bar{\delta}_V}{\tau_0} \to 0 \quad (V\to\infty)$$

uniformly with respect to the temperature θ in any interval of the form $(0<\theta\leqslant\theta_0)$.

The validity of paragraph 3 of conditions I is thereby also established.

The remaining paragraphs of conditions I and I′ follow trivially from the inequalities (3.37) and (3.38), the condition (3.39), the fact that s is finite in the sums over α, and the fact that the quantities C are independent of V.

We can therefore make use of all the limit theorems proved in Chapters 1 and 2.

Since $\Gamma_a = H_\infty(\bar{C})$, we write the theorems on the existence of the limits

$$\lim_{V\to\infty} \langle \mathcal{A} \rangle_\Gamma = \lim_{V\to\infty} \langle \mathcal{A} \rangle_{\Gamma_a}$$

in the form

$$\lim_{V\to\infty} \langle \mathcal{A} \rangle_\Gamma = \lim_{V\to\infty} \langle \mathcal{A} \rangle_{H(\bar{C})}.$$

But $H(C)$ is independent of the parameters τ in the case under consideration when

$$0 < \tau_\alpha < 1, \quad \alpha = 1, 2, \ldots, s. \tag{3.87}$$

Therefore, the expression

$$\lim_{V\to\infty} \langle \mathcal{A} \rangle_\Gamma,$$

is also independent of the τ lying in the region (3.87).

Consequently, when all the $\tau_1, \ldots, \tau_s$ tend to zero while remaining positive, we have, trivially

$$\lim_{\tau\to 0} \lim_{V\to\infty} \langle \mathcal{A} \rangle_\Gamma = \lim_{V\to\infty} \langle \mathcal{A} \rangle_{H(\bar{C})}.$$

We can define the quasi-average in this situation by the relations

$$\prec \mathcal{A} \succ_H = \lim_{V\to\infty} \langle \mathcal{A} \rangle_\Gamma = \lim_{V\to\infty} \langle \mathcal{A} \rangle_{H(\bar{C})} \tag{3.88}$$

in which the τ can take any values from the range (3.87).

We emphasize again that the most important point in our arguments was the establishment of the inequality (3.82), based on the inequality (3.81).

It can be seen that from the inequality (3.81) there follows the asymptotic relation

$$\lim_{V\to\infty} f_V(\Gamma) = \lim_{V\to\infty} f_V(H). \tag{3.89}$$

§ 9. THE QUESTION OF THE CHOICE OF SIGN FOR THE SOURCE-TERMS

We note, in passing, that the above asymptotic relation ceases, in general, to be true for negative values of τ. In fact, we take, for example,

$$\tau_\alpha = -\omega_\alpha; \qquad \omega_\alpha > 0 \quad (\alpha = 1, 2, \ldots, s). \tag{3.90}$$

Then

$$\Gamma = \Gamma_\omega = H - 2V\sum_\alpha \omega_\alpha g_\alpha (J_\alpha - C_\alpha)(J_\alpha^+ - C_\alpha^*).$$

We make use of the inequality (3.63), substituting in it

$$\Gamma_0 = \Gamma, \qquad \Gamma_1 = 2V\sum_\alpha \omega_\alpha g_\alpha (J_\alpha - C_\alpha)(J_\alpha^+ - C_\alpha^*).$$

Then in (3.63) the Hamiltonian will be

$$\Gamma_0 + \Gamma_1 = H$$

and

$$f_V(H) - f_V(\Gamma_\omega) \geqslant 2\sum_\alpha \omega_\alpha g_\alpha \langle (J_\alpha - C_\alpha)(J_\alpha^+ - C_\alpha^*)\rangle_H . \tag{3.91}$$

But, obviously,

$$\langle a_f^+ a_{-f}^+ \rangle_H = \langle a_{-f} a_f \rangle_H = 0,$$

and therefore

$$\langle J_\alpha \rangle_H = 0, \quad \langle J_\alpha^+ \rangle_H = 0.$$

We have, consequently,

$$\langle (J_\alpha - C_\alpha)(J_\alpha^+ - C_\alpha^*)\rangle_H = \langle J_\alpha J_\alpha^+ \rangle_H + |C_\alpha|^2,$$

whence, by virtue of (3.91), we find:

$$f_V(H) - f_V(\Gamma_\omega) \geqslant 2\sum_\alpha \omega_\alpha g_\alpha |C_\alpha|^2,$$

and, passing to the limit,

$$\lim_{V\to\infty} f_V(H)-\lim_{V\to\infty} f_V(\Gamma_\omega) \geqslant 2\sum_\alpha \omega_\alpha g_\alpha |\bar{C}_\alpha|^2. \tag{3.92}$$

Thus, if $\bar{C} \neq 0$, the equality (3.89) is not true for negative τ (3.90).

It was to take this fact into account that we required that the proportionality coefficients r_α in our choice (3.75) of parameters ν_α characterizing the sources included in the Hamiltonian be positive.

§ 10. THE CONSTRUCTION OF UPPER-BOUND INEQUALITIES IN THE CASE WHEN $\bar{C} = 0$

We give here a separate treatment of the case when $\bar{C} = 0$, i.e. when

$$\bar{C}_1 = \bar{C}_2 = \ldots = \bar{C}_s = 0. \tag{3.93}$$

In this case,

$$H(\bar{C}) = T,$$

and, therefore,

$$f_V(H)-f_V(T) \to 0 \quad \text{as} \quad V \to \infty.$$

Thus, the interaction terms

$$-2V \sum_{1\leqslant\alpha\leqslant s} g_\alpha J_\alpha J_\alpha^+$$

of the Hamiltonian H are asymptotically (as $V \to \infty$) ineffective in the calculation of the free energy.

Further, we have (compare with (3.78))

$$\Gamma = \Gamma_\tau = H+2V\sum_\alpha g_\alpha \tau_\alpha J_\alpha J_\alpha^+ = T-2V\sum_\alpha g_\alpha(1-\tau_\alpha) J_\alpha J_\alpha^+, \tag{3.94}$$

and, in view of our earlier proofs, we can write an upper bound for a correlation average constructed on the basis of this Hamiltonian:

$$\left\langle \sum_\alpha g_\alpha J_\alpha J_\alpha^+ \right\rangle_\Gamma \leqslant \frac{\delta_V+\bar{\delta}_V}{2\tau_0} \to 0 \quad \text{as} \quad V \to \infty. \tag{3.95}$$

We shall show that, in the given case (3.93), we have also

$$\left\langle \sum_\alpha g_\alpha J_\alpha J_\alpha^+ \right\rangle_H \leqslant \zeta_V \to 0 \quad (V \to \infty). \tag{3.96}$$

For this, we shall take the Hamiltonian

$$H_\omega = T - 2V(1+\omega)\sum_\alpha g_\alpha J_\alpha J_\alpha^+ \qquad (1 > \omega > 0), \tag{3.97}$$

and formulate a trial Hamiltonian

$$\begin{aligned} H_\omega(C) = T - 2V(1+\omega)\sum_\alpha g_\alpha(C_\alpha J_\alpha^+ + C_\alpha^* J_\alpha) \\ + 2V(1+\omega)\sum_\alpha g_\alpha C_\alpha^* C_\alpha . \end{aligned} \tag{3.98}$$

We denote by $\bar{C}^{(\omega)}$ the point C giving the absolute minimum of the function $f_\infty\{H_\omega(C)\}$. If for any positive ω, however small,

$$\bar{C}^{(\omega)} = 0, \tag{3.99}$$

the proof of the relation (3.96) is trivial. We need only replace H by H_ω in the inequality (3.95) and for τ_α take

$$\tau_\alpha = \frac{\omega}{1+\omega}$$

in the Hamiltonian Γ_τ. Then the Hamiltonian Γ in (3.95) coincides with H. It remains, therefore, to consider the case when (3.99) is not true for some positive value of ω, however small.

We note that the value $C = \bar{C}^{(\omega)}$ must satisfy the equations

$$\frac{\partial f_\infty\{H_\omega(C)\}}{\partial C_\alpha^*} = 0 \qquad (1 \leqslant \alpha \leqslant s);$$

i.e., from (3.54),

$$\bar{C}_\alpha^{(\omega)} = \frac{1}{2(2\pi)^3}\int \frac{\tanh[E_\omega(f)/2\theta]}{E_\omega(f)} \left\{\sum_\beta (1+\omega) g_\beta \bar{C}_\beta^{(\omega)} \lambda_\beta^*(f)\right\} \lambda_\alpha(f)\, df,$$

where

$$E_\omega^2(f) = \left(\frac{p^2}{2m} - \mu\right)^2 + 4(1+\omega)^2 \left|\sum_\beta g_\beta \bar{C}_\beta^{(\omega)} \lambda_\beta^*(f)\right|^2 .$$

Hence, it follows that

$$|\bar{C}_\alpha^{(\omega)}| \leqslant \frac{1}{4(2\pi)^3}\int |\lambda_\alpha(f)|\, df \leqslant \frac{\bar{Q}_1}{2} . \tag{3.100}$$

We now put

$$(1+\omega)C_\alpha = X_\alpha, \tag{3.101}$$

and note that

$$f_\infty\{H_\omega(C)\} = f_\infty\{H(X)\} - 2\frac{\omega}{1+\omega}\sum_\alpha g_\alpha |X_\alpha|^2. \tag{3.102}$$

Thus, for

$$X_\alpha^{(\omega)} = (1+\omega)\bar{C}_\alpha^{(\omega)},$$

the expression on the right-hand side of (3.102) attains an absolute minimum.

Therefore

$$f_\infty\{H(X^{(\omega)})\} - 2\frac{\omega}{1+\omega}\sum_\alpha g_\alpha |X_\alpha^{(\omega)}|^2 \leqslant f_\infty\{H(0)\}. \tag{3.103}$$

On the other hand,

$$f_\infty\{H(0)\} = \min_{(X)} f_\infty\{H(X)\},$$

in view of which,

$$f_\infty\{H(X^{(\omega)})\} \geqslant f_\infty\{H(0)\}.$$

Hence,

$$0 \leqslant f_\infty\{H(X^{(\omega)})\} - f_\infty\{H(0)\} \leqslant 2\frac{\omega}{1+\omega}\sum_\alpha g_\alpha |X_\alpha^{(\omega)}|^2. \tag{3.104}$$

We shall show now that

$$X^{(\omega)} \to 0 \quad (\omega \to 0). \tag{3.105}$$

In fact, we assume the opposite to be the case. Then, since, by virtue of (3.100), $X^{(\omega)}$ is bounded:

$$|X_\alpha^{(\omega)}| \leqslant (1+\omega)\frac{\bar{Q}_1}{2} \leqslant \bar{Q}_1,$$

we can always choose a sequence of positive $\omega' \to 0$ such that

$$X^{(\omega')} \to \mathring{X},$$

with

$$\mathring{X} \neq 0. \tag{3.106}$$

Putting $\omega = \omega'$ in (3.104) and passing to the limit, we find

$$f_\infty\{H(\tilde{X})\} = f_\infty\{H(0)\}. \tag{3.107}$$

But, as we have seen, if the point $C = 0$ gives the absolute minimum of the function $f_\infty\{H(C)\}$, no other point realizing the absolute minimum of this function exists; in view of this, (3.107) and (3.106) are inconsistent and we have arrived at a contradiction.

Thus, the relation (3.105) is proved. Noting that

$$f_\infty\{H(0)\} = f_\infty(T),$$

from (3.102) and (3.104) we obtain

$$-2\frac{\omega}{1+\omega}\sum_\alpha g_\alpha |X_\alpha^{(\omega)}|^2 \leqslant f_\infty\{H_\omega(\bar{C}^{(\omega)})\} - f_\infty(T) \leqslant 0;$$

i.e.,

$$0 \leqslant f_\infty(T) - f_\infty\{H_\omega(\bar{C}^{(\omega)})\} \leqslant \omega\xi(\omega), \tag{3.108}$$

where

$$\xi(\omega) = \frac{2}{1+\omega}\sum_\alpha g_\alpha |X_\alpha^{(\omega)}|^2 \to 0 \quad \text{as} \quad \omega \to 0. \tag{3.109}$$

We now invoke Theorem 3.III. Since, in the case under consideration, $H(\bar{C}) = H(0) = T$, we can write

$$|f_V(H) - f_\infty(T)| \leqslant \delta_V \to 0, \quad (V \to \infty).$$

For the Hamiltonian H_ω, we also have

$$|f_V(H_\omega) - f_\infty\{H_\omega(\bar{C}^{(\omega)})\}| \leqslant \delta_V(\omega) \to 0, \quad \text{as} \quad V \to \infty.$$

Here, $\delta_V(\omega)$ denotes δ_V for H_ω. On the other hand,

$$\begin{aligned} 0 \leqslant f_V(H) - f_V(H_\omega) &= f_V(H) - f_\infty(T) + f_\infty(T) \\ &\quad - f_\infty\{H_\omega(\bar{C}^{(\omega)})\} + f_\infty\{H_\omega(\bar{C}^{(\omega)})\} - f_V(H_\omega), \end{aligned}$$

and, therefore,

$$0 \leqslant f_V(H) - f_V(H_\omega) \leqslant \delta_V + \delta_V(\omega) + \omega\xi(\omega). \tag{3.110}$$

We shall now make use of the inequality (3.63); we substitute in it

$$\Gamma_0 = H_\omega, \quad \Gamma_1 = 2V\omega \sum_\alpha g_\alpha J_\alpha J_\alpha^+,$$
$$\Gamma_0 + \Gamma_1 = H.$$

Then, from (3.110) we obtain

$$\left\langle \sum_\alpha g_\alpha J_\alpha J_\alpha^+ \right\rangle_H \leqslant \frac{\delta_V + \delta_V(\omega)}{2\omega} + \frac{1}{2}\xi(\omega). \tag{3.111}$$

However, this inequality is valid for any value of ω in the interval $(0 < \omega < 1)$, and its left-hand side is entirely independent of ω. Consequently, the left-hand side of (3.111) will not exceed the lower bound of the right-hand side in the given interval:

$$\left\langle \sum_\alpha g_\alpha J_\alpha J_\alpha^+ \right\rangle_H \leqslant \zeta_V,$$
$$\zeta_V = \inf_{(0<\omega<1)} \left\{ \frac{\delta_V + \delta_V(\omega)}{2\omega} + \frac{1}{2}\xi(\omega) \right\}.$$

It only remains for us to show that ζ_V goes to zero as $V \to \infty$.

We shall fix an arbitrarily small number ϱ. On the basis of (3.109), we can, in the interval $(0 < \omega < 1)$ under consideration, fix a number ω_0 such that

$$\xi(\omega_0) \leqslant \varrho.$$

We see then that

$$\zeta_V \leqslant \frac{\delta_V + \delta_V(\omega_0)}{2\omega_0} + \frac{\varrho}{2}.$$

But, since ω_0 is fixed, we have

$$\frac{\delta_V + \delta_V(\omega_0)}{2\omega_0} \to 0 \quad \text{as} \quad V \to \infty.$$

We can find a value V_0 such that

$$\frac{\delta_V + \delta_V(\omega_0)}{2\omega_0} \leqslant \frac{\varrho}{2} \quad \text{for} \quad V \geqslant V_0.$$

Thus,

$$\zeta_V \leqslant \varrho \quad \text{for} \quad V \geqslant V_0;$$

i.e., $\zeta_V \to 0$ as $V \to \infty$, and the relation (3.96) is proved.

It is now clear that H is of the type (1.14), (1.15), with

$$\Gamma_a = T, \quad \Lambda(f) = 0, \quad C_\alpha = 0, \quad G_\alpha = 2g_\alpha.$$

By virtue of (3.96), paragraph 3 of conditions I (Chapter 1, § 1) is fulfilled; the remaining paragraphs of conditions I and I′ (Chapter 2, § 7) are trivial in the given case.

We shall, therefore, make use of the limit theorems proved in Chapters 1 and 2. We see that, for the operators $\mathcal{A}$ with which these theorems are concerned, we can write

$$\lim_{V\to\infty} \langle \mathcal{A} \rangle_H = \lim_{V\to\infty} \langle \mathcal{A} \rangle_T .$$

Applying (3.88), we see that, in the case under investigation ($C = 0$), we have

$$\prec \mathcal{A} \succ_H = \lim_{V\to\infty} \langle \mathcal{A} \rangle_H = \lim_{V\to\infty} \langle \mathcal{A} \rangle_T .$$

Thus, the quasi-average and the usual average of the operators considered above are asymptotically equal to the corresponding averages taken over the Hamiltonian T. Moreover, the interaction terms in H turn out to have *no effect* here.

Footnote to p. 92 *(added in proof)*. Mathematical questions concerning the existence and analyticity properties of partition functions have been considered in a paper by the author (*Physica* **41**, 601 (1969)). The theorem proved there has been further generalized by H. D. Maison in Preprint Th. 1299-CERN (Geneva), 25 Feb. 1971.

CHAPTER 4

MODEL SYSTEMS WITH POSITIVE AND NEGATIVE INTERACTION COMPONENTS

IN THIS chapter we consider model systems with positive interaction components and also systems containing both positive and negative four-fermion interaction components. We construct, under the conditions I of Chapter 1, § 1, an asymptotically exact solution for such model systems.

The technique developed here leads to a minimax principle for model problems with four-fermion interaction, and enables us to find asymptotically exact expressions for the free energies and many-time correlation averages

A second, equivalent, formulation of the minimax principle is also proposed.

§ 1. HAMILTONIAN WITH NEGATIVE COUPLING CONSTANTS (REPULSIVE INTERACTION)

We shall consider the Hamiltonian obtained by replacing g_α by $-g_\alpha$ in the Hamiltonian studied in the previous chapter:

$$H = T+2V \sum_{1\leq\alpha\leq r} g_\alpha J_\alpha J_\alpha^+ , \tag{4.1}$$

$$g_\alpha > 0, \quad \alpha = 1, \ldots, r.$$

Reasoning by analogy, we shall attempt to construct, for this H, the usual form for the corresponding trial Hamiltonian

$$H(C) = T+2V\sum_{\alpha} g_\alpha(C_\alpha J_\alpha^+ + C_\alpha^* J_\alpha - C_\alpha C_\alpha^*). \tag{4.2}$$

We have

$$H = H(C)+2V\sum_{\alpha} g_\alpha(J_\alpha - C_\alpha)(J_\alpha^+ - C_\alpha^*) \geqslant H(C).$$

Now, therefore, in contrast to the preceding chapter

$$f_V(H) \geqslant f_V\{H(C)\}. \tag{4.3}$$

This inequality is fulfilled for any complex C. Consequently, if we wish to select the constants C in (4.2) in such a way that $f_V\{H(C)\}$ be as close as possible to the expression for $f_V(H)$, we must determine C from the ***absolute maximum*** condition

$$\begin{gathered} C = \bar{C}, \\ f_V\{H(\bar{C})\} = \max_{(C)} f_V\{H(C)\}. \end{gathered} \tag{4.4}$$

We shall assume now that the operators T and J_α in the Hamiltonian (4.1) under consideration satisfy the conditions

$$\begin{gathered} |f_V(T)| \leqslant M_0, \quad |J_\alpha| \leqslant M_1, \\ M_0,\ M_1 = \text{const}. \end{gathered} \tag{4.5}$$

We introduce the expression

$$\begin{aligned} F_V(C) &= f_V\{H(C)\}+2\sum_{\alpha} g_\alpha C_\alpha C_\alpha^* \\ &= f_V\left\{T+2V\sum_{\alpha} g_\alpha(C_\alpha J_\alpha^+ + C_\alpha^* J_\alpha)\right\} \end{aligned} \tag{4.6}$$

and note that

$$\frac{\partial F_V(C)}{\partial C_\alpha} = 2g_\alpha\langle J_\alpha^+\rangle_{H(C)}, \qquad \frac{\partial F_V(C)}{\partial C_\alpha^*} = 2g_\alpha\langle J_\alpha\rangle_{H(C)}.$$

Therefore, by virtue of the conditions (4.5),

$$\left|\frac{\partial F_V(C)}{\partial C_\alpha}\right| \leqslant 2g_\alpha M_1, \qquad \left|\frac{\partial F_V(C)}{\partial C_\alpha^*}\right| \leqslant 2g_\alpha M_1. \tag{4.7}$$

On the other hand, the problem of the absolute maximum of the function

$$f_V\{H(C)\} \tag{4.8}$$

is equivalent to the problem of the absolute minimum of the function

$$-f_V\{H(C)\} = -F_V(C) + 2\sum_\alpha g_\alpha C_\alpha^* C_\alpha .$$

Therefore, taking the inequalities (4.7) into account and repeating the arguments from the beginning of the previous chapter, we convince ourselves of the existence of quantities $C = \bar{C}$ realizing the absolute maximum of the function (4.8) in the space of all the points C $(C_1, \ldots, C_s)$ (with complex values C_α).

Since the function (4.8) possesses continuous partial derivatives of all orders with respect to the variables $\ldots C_\alpha \ldots C_\alpha^* \ldots$, we see that the values

$$C_\alpha = \bar{C}_\alpha, \qquad C_\alpha^* = \bar{C}_\alpha^*, \qquad (\alpha = 1, \ldots, r)$$

are solutions of the equations

$$\frac{\partial f_V\{H(C)\}}{\partial C_\alpha} = 0, \qquad \frac{\partial f_V\{H(C)\}}{\partial C_\alpha^*} = 0. \tag{4.9}$$

Hence,

$$\bar{C}_\alpha = \langle J_\alpha \rangle_{H(\bar{C})}, \qquad \bar{C}_\alpha^* = \langle J_\alpha^+ \rangle_{H(\bar{C})}. \tag{4.10}$$

Consequently, on the basis of (4.5),

$$|\bar{C}_\alpha| \leqslant M_1, \qquad \alpha = 1, \ldots, r. \tag{4.11}$$

We note that in the case of the model system (4.1) the point $C = \bar{C}$ is unique. Moreover, we shall show now that the solution of equations (4.9) is *unique*. For this, we construct the expression

$$f_V\{H(C+tz)\} = F_V(C+tz) - 2\sum_\alpha g_\alpha (C_\alpha^* + t z_\alpha^*)(C_\alpha + t z_\alpha), \qquad z = (z_1, \ldots, z_r). \tag{4.12}$$

(z_α is an arbitrary complex number.) But

$$F_V(C+tz) = f_V(\Gamma_0+t\Gamma_1),$$

where

$$\Gamma_0 = T+2V\sum_\alpha g_\alpha(C_\alpha J_\alpha^+ + C_\alpha^* J_\alpha),$$

$$\Gamma_1 = 2V\sum_\alpha g_\alpha(z_\alpha J_\alpha^+ + z_\alpha^* J_\alpha),$$

and, therefore, by virtue of (3.62),

$$\frac{d^2}{dt^2} F_V(C+tz) \leqslant 0. \tag{4.13}$$

Hence, it follows from (4.12) that

$$\frac{d^2 f_V\{H(C+tz)\}}{dt^2} \leqslant -4\sum_\alpha g_\alpha z_\alpha^* z_\alpha . \tag{4.14}$$

It is not difficult to see that the equations (4.9) have the solution which gives the function (4.8) its absolute maximum.

Let the equations (4.9) also have some other solution $C = \tilde{C}$. We put

$$z = \tilde{C}-\bar{C}, \quad C = \bar{C}.$$

Then, with these values of z,

$$\frac{df_V\{H(\bar{C}+tz)\}}{dt} = 0 \quad \text{for} \quad t = 0 \quad \text{and} \quad t = 1,$$

whence

$$\int_0^1 \frac{d^2 f_V\{H(\bar{C}+tz)\}}{dt^2}\, dt = 0$$

and, therefore, by virtue of (4.14),

$$\sum_\alpha g_\alpha z_\alpha^* z_\alpha = 0;$$

i.e., $z = 0$.

Thus, we have proved that the solution of equations (4.9) is unique and gives the absolute maximum of the function (4.8). Finally, we shall make use of the inequality (3.34), in which we put:

$$\Gamma_0 = H(\overline{C}), \quad \Gamma_1 = 2V\sum_{\alpha} g_\alpha (J_\alpha - \overline{C}_\alpha)(J_\alpha^+ - \overline{C}_\alpha^*),$$

$$\Gamma_0 + \Gamma_1 = H.$$

We obtain:

$$\begin{aligned} 2\sum_{\alpha} g_\alpha \langle (J_\alpha - \overline{C}_\alpha)(J_\alpha^+ - \overline{C}_\alpha^*)\rangle_H &\leqslant f(H) - f\{H(\overline{C})\} \\ &\leqslant 2\sum_{\alpha} g_\alpha \langle (J_\alpha - \overline{C}_\alpha)(J_\alpha^+ - \overline{C}_\alpha^*)\rangle_{H(C)}\,. \end{aligned} \tag{4.15}$$

After this preliminary discussion of the properties of the form of the trial Hamiltonian, we return to our consideration of the Hamiltonian (4.1) itself.

§ 2. FEATURES OF THE ASYMPTOTIC RELATIONS FOR THE FREE ENERGIES IN THE CASE OF SYSTEMS WITH POSITIVE INTERACTION

Here the question arises as to whether a theorem analogous to Theorem 3.I is true for a Hamiltonian of the type (4.1).

In other words, can we assert that

$$f_V(H) - f_V\{H(\overline{C})\} \to 0 \quad \text{as} \quad V \to \infty \tag{4.16}$$

when the operators T and J_α occurring in (4.1) satisfy the conditions (3.30) and (3.31)? We shall show now that such an assertion is *not true* even in the simplest case $r = 1$, when

$$H = T + 2VgJJ^+\,. \tag{4.17}$$

For this, we shall consider an operator J of the form

$$J = \frac{1}{2V}\sum_{f} \lambda(f) a_f^+ a_{-f}^+, \tag{4.18}$$

where

$$\begin{gathered} \lambda(f) = \lambda(p, \sigma) = \begin{cases} \lambda_0(\sigma), & \left|\dfrac{p^2}{2m} - \mu\right| \leqslant \varDelta, \\ 0, & \left|\dfrac{p^2}{2m} - \mu\right| > \varDelta, \end{cases} \\ \lambda_0(\sigma) = \begin{cases} \tilde{\lambda}, & \sigma = \frac{1}{2}, \\ -\tilde{\lambda}, & \sigma = -\frac{1}{2}; \end{cases} \qquad \begin{aligned} \tilde{\lambda} &= \text{const} > 0, \\ \varDelta &= \text{const} > 0. \end{aligned} \end{gathered} \tag{4.19}$$

For the operator T in (4.17), we take

$$T = H_0 = T_0 - 2VgJJ^+,$$
$$T_0 = \sum_f \left(\frac{p^2}{2m} - \mu\right) a_f^\dagger a_f. \tag{4.20}$$

This Hamiltonian H_0 obviously belongs, by virtue of (4.18), (4.19) and (4.20), to the class of Hamiltonians considered in the previous chapter, and satisfies the conditions of Theorem 3.III.

Here, we choose the values of g, λ and θ in such a way that the quantity $S = \bar{S}$ which realizes the absolute minimum

$$\min_{(S)} f_\infty\{H_0(S)\} = f_\infty\{H_0(\bar{S})\}$$

is non-zero. Then

$$f_\infty\{H_0(\bar{S})\} < f_\infty\{H_0(0)\} = f_\infty(T_0),$$

and, on the basis of Theorem 3.III,

$$\lim_{V\to\infty} f_V(H_0) = f_\infty\{H_0(\bar{S})\}.$$

Consequently, we have

$$\lim_{V\to\infty} \{f_V(T_0) - f_V(T)\} > 0. \tag{4.21}$$

On the other hand, by virtue of (4.18), (4.19) and (4.20), the operators T and J in (4.17) satisfy the conditions (3.30) and (3.31).

Therefore, if the assertion (4.16) were true for the model system (4.17), we would be able to write

$$f_V(T_0) - f_V\{H(\bar{C})\} \to 0 \quad (V \to \infty), \tag{4.22}$$

since, in the present case, $H = T_0$ identically.

We shall now find $\bar{C}$. We have

$$H(C) = T + 2Vg(CJ^+ + C^*J - C^*C).$$

As has been pointed out, the value $C = \bar{C}$ is determined from the equations

$$\frac{\partial f_V\{H(C)\}}{\partial C} = 0, \quad \frac{\partial f_V\{H(C)\}}{\partial C^*} = 0; \tag{4.23}$$

i.e.,

$$\bar{C} = \langle J \rangle_{H(\bar{C})}, \quad \bar{C}^* = \langle J^+ \rangle_{H(\bar{C})}.$$

It is clear that these equations have the trivial solution $C = C^* = 0$, since, by virtue of (4.18), the identities

$$\langle J \rangle_{H(0)} = \langle J \rangle_T = 0,$$
$$\langle J^+ \rangle_{H(0)} = \langle J^+ \rangle_T = 0$$

are true. Since the equations (4.23) have a unique solution, we have

$$\bar{C} = 0, \quad H(\bar{C}) = T.$$

Therefore, from (4.22) we obtain

$$\lim_{V \to \infty} \{f_V(T_0) - f_V(T)\} = 0,$$

which contradicts (4.21). Thus, in contrast to the case of negative interaction (3.29), in the positive-interaction case (4.1) under consideration the asymptotic relation (4.16) can be false even when the conditions (3.30) and (3.31) are fulfilled.

The absence of an analogue of Theorem 3.I, which served as the basis of our investigations in the preceding chapter, can, generally speaking, make the study of the model Hamiltonian (4.1) in the general case more difficult.

However, when the operators T and J_α have the special form (3.2), we obtain an even simpler situation than in the preceding chapter.

§ 3. BOUNDS FOR THE FREE ENERGIES AND CORRELATION FUNCTIONS

Let the operators T and J_α occurring in (4.1) be represented by the formulae (3.2). We shall also assume that the functions

$$\lambda_\alpha(f) = \lambda_\alpha(p, \sigma)$$

satisfy the following conditions:

$$\left.\begin{array}{ll}\text{(a)} & \frac{1}{V}\sum_{p}|\lambda_\alpha(p,\sigma)| \leqslant Q_1 = \text{const}\\ \text{(b)} & |\lambda_\alpha(p,\sigma)| \leqslant \bar{Q} = \text{const,}\\ \text{(c)} & \text{The discontinuities of the functions } \lambda_\alpha(p,\sigma) \text{ form a}\\ & \text{set of measure zero in the space } E.\end{array}\right\} \quad (4.24)$$

We note, in passing, that, in particular, it follows from these conditions that

$$\frac{1}{V}\sum_{p}|\lambda_\alpha(p,\sigma)|^2 \leqslant Q_2 = \text{const} \qquad (4.25)$$

where as Q_2 we could take $Q_2 = \bar{Q}Q_1$.

We shall consider the equations (4.9) for $\bar{C}$ and write them in the form (4.10):

$$\bar{C}_\alpha = \langle J_\alpha \rangle_{H(\bar{C})}, \quad \bar{C}_\alpha^* = \langle J_\alpha^+ \rangle_{H(\bar{C})}.$$

Since

$$\langle J_\alpha \rangle_T = \langle J_\alpha^+ \rangle_T = 0$$

identically and $H(0) = T$, we see again that the equations under consideration have the solution

$$\bar{C}_\alpha = 0, \quad \bar{C}_\alpha^* = 0,$$

and, by virtue of our earlier proof, this solution is unique.

Thus,

$$H(\bar{C}) = H(0) = T.$$

We now make use of the inequalities (4.15) and obtain

$$0 \leqslant f_V(H) - f_V(T) \leqslant 2\sum_\alpha g_\alpha \langle J_\alpha J_\alpha^+ \rangle_T, \qquad (4.26)$$

$$2\sum_\alpha g_\alpha \langle J_\alpha J_\alpha^+ \rangle_H \leqslant f_V(H) - f_V(T). \qquad (4.27)$$

We shall write out the right-hand side of inequality (4.26) explicitly. We have

$$\langle J_\alpha J_\alpha^+ \rangle_T = \frac{1}{4V^2}\sum_{f,f'} \lambda_\alpha(f)\,\lambda_\alpha^*(f')\,\langle a_f^+ a_{-f}^+ a_{-f'} a_{f'} \rangle_T.$$

But the average

$$\langle a_f^+ a_{-f}^+ a_{-f'} a_{f'} \rangle_T$$

can be non-zero only if $f' = f$ or $f' = -f$. Further,

$$\langle a_f^+ a_{-f}^+ a_{-f} a_f \rangle_T \leq 1$$

and

$$|\langle a_f^+ a_{-f}^+ a_f a_{-f} \rangle_T| = |\langle a_f^+ a_f a_{-f}^+ a_{-f} \rangle_T| \leq 1.$$

Therefore,

$$\langle J_\alpha J_\alpha^+ \rangle_T \leq \frac{1}{2V^2} \sum_f |\lambda_\alpha(f)|^2 \leq \frac{1}{V} Q_2,$$

whence, from (4.26) and (4.27), we find

$$0 \leq f_V(H) - f_V(T) \leq \frac{2}{V} Q_2 \sum_\alpha g_\alpha, \tag{4.28}$$

$$\left\langle \sum_\alpha g_\alpha J_\alpha J_\alpha^+ \right\rangle_H \leq \frac{1}{V} Q_2 \sum_\alpha g_\alpha. \tag{4.29}$$

Thus, the interaction terms

$$\sum_\alpha g_\alpha J_\alpha J_\alpha^+$$

contained in H have, asymptotically ($V \to \infty$), no effect on the free energy:

$$f_V(H) - f_V(T) \to 0 \quad \text{as} \quad V \to \infty.$$

Furthermore, it is not difficult to see that H reduces to the form (1.14), (1.15), if in the latter we put

$$\Gamma = H, \quad \Gamma_a = T, \quad C_\alpha = 0, \quad \Lambda = 0, \quad G_\alpha = -2g_\alpha.$$

By virtue of (4.29), paragraph 3 of conditions I of Chapter 1, § 1 is fulfilled. The remaining paragraphs of conditions I and I′ can be verified trivially starting from (4.24).

We can therefore use the limit theorems of Chapters 1 and 2.

It follows from these that, in the given situation, averages

$$\langle \mathcal{A} \rangle_H$$

of the operators $\mathcal{A}$ to which these theorems apply are asymptotically equal to the corresponding averages

$$\langle \mathcal{A} \rangle_T$$

taken over the Hamiltonian T, so that the interaction terms in H are also asymptotically ineffective here.

In the case being considered, the quasi-averages do not differ from the asymptotic values of the usual averages, inasmuch as they are defined in terms of the Hamiltonian

$$\Gamma_\tau = T + 2V \sum_\alpha g_\alpha (1+\tau_\alpha) J_\alpha J_\alpha^+ ,$$

which has the same form as H, except with the transformed coupling constants: $g_\alpha \to g_\alpha(1+\tau_\alpha)$; in view of this, averages of the operators $\mathcal{A}$ taken over the Hamiltonian Γ_τ are asymptotically equal to the corresponding averages over T, provided that $\tau_\alpha > -1$.

§ 4. EXAMINATION OF AN AUXILIARY PROBLEM

We shall consider a model Hamiltonian of rather more general form, namely:

$$H = T_0 + 2V \sum_{1 \leq \beta \leq l} \{U_\beta I_\beta^+ + U_\beta^* I_\beta\} + 2V \sum_{1 \leq \alpha \leq r} g_\alpha J_\alpha J_\alpha^+ \qquad (g_\alpha > 0), \tag{4.30}$$

where

$$T_0 = \sum_f \left(\frac{p^2}{2m} - \mu \right) a_f^+ a_f ,$$

$$J_\alpha = \frac{1}{2V} \sum_f \lambda_\alpha(f) a_f^+ a_{-f}^+, \qquad I_\beta = \frac{1}{2V} \sum_f \mu_\beta(f) a_f^+ a_{-f}^+, \tag{4.31}$$

$$\lambda_\alpha(-f) = -\lambda_\alpha(f), \quad \mu_\beta(-f) = -\mu_\beta(f),$$

and it is assumed that the functions $\lambda_\alpha(f)$ and $\mu_\beta(f)$ satisfy the conditions (4.24).

Furthermore, here U_β are complex parameters which can take any finite values.

It is clear that a Hamiltonian of the type (4.30) reduces to the form (4.1) if in the latter we put

$$T = T_0 + 2V \sum_\beta \{U_\beta I_\beta^+ + U_\beta^* I_\beta\}. \tag{4.32}$$

It is easily verified that the conditions (4.5) are fulfilled, with, in this case,

$$M_1 = Q_1.$$

Therefore, the form of the trial Hamiltonian will be

$$\begin{aligned} H(C) = T_0 + 2V \sum_\beta \{U_\beta I_\beta^+ + U_\beta^* I_\beta\} \\ + 2V \sum_\alpha g_\alpha \{C_\alpha J_\alpha^+ + C_\alpha^* J_\alpha - C_\alpha C_\alpha^*\}. \end{aligned} \tag{4.33}$$

The quantity C_α is determined from the equations

$$\frac{\partial f_V\{H(C)\}}{\partial C_\alpha} = 0, \quad \frac{\partial f_V\{H(C)\}}{\partial C_\alpha^*} = 0. \tag{4.34}$$

According to our earlier proofs, these equations have a single unique solution. This solution gives the absolute maximum of the function $f_V\{H(C)\}$ in the space of all the points C.

Since the expression (4.32) contains terms I_β and I_β^+, averages $\langle J_\alpha \rangle_T$ taken over the Hamiltonian T (4.32) are no longer necessarily trivially zero; thus, the solution of the equations (4.34) can be non-zero. Since this solution depends in general on V, we shall denote it by

$$C = \bar{C}^{(V)}.$$

In accordance with (4.11),

$$|\bar{C}_\alpha^{(V)}| \leqslant Q_1, \quad \alpha = 1, 2, \ldots, r. \tag{4.35}$$

For brevity, we put

$$H\{\bar{C}^{(V)}\} = H_a, \tag{4.36}$$

and write, using (4.15),

$$0 \leqslant f_V(H) - f_V(H_a) \leqslant 2 \sum_\alpha g_a \langle (J_a - \bar{C}_\alpha^{(V)})(J_\alpha^+ - \bar{C}^{*(V)}) \rangle_{H_a}. \tag{4.37}$$

We shall now write out the right-hand side of this inequality explicitly.

It follows from (4.34) that

$$\bar{C}_\alpha^{(V)} = \langle J_\alpha \rangle_{H_a}, \quad \bar{C}^{*(V)} = \langle J_\alpha^+ \rangle_{H_a},$$

and therefore we have

$$\langle (J_\alpha - \bar{C}_\alpha^{(V)})(J_\alpha^+ - \bar{C}_\alpha^{*(V)}) \rangle_{H_a} = \langle J_\alpha J_\alpha^+ \rangle_{H_a} - \langle J_\alpha \rangle_{H_a} \langle J_\alpha^+ \rangle_{H_a}.$$

Consequently,

$$\begin{aligned} &\langle (J_\alpha - \bar{C}_\alpha^{(V)})(J_\alpha^+ - \bar{C}_\alpha^{*(V)}) \rangle_{H_a} \\ &\quad = \frac{1}{4V^2} \sum_{f,f'} \lambda_\alpha(f)\, \lambda_\alpha^*(f') \left\{ \langle a_f^+ a_{-f}^+ a_{-f'} a_{f'} \rangle_{H_a} - \langle a_f^+ a_{-f}^+ \rangle_{H_a} \langle a_{-f'} a_{f'} \rangle_{H_a} \right\}. \end{aligned} \tag{4.38}$$

Since the Hamiltonian H_a (see (4.36)) is a quadratic form in the Fermi operators, we can use the technique of Bloch and de Dominicis to expand the expression $\langle a_f^+ a_{-f}^+ a_{-f'} a_{f'} \rangle_{H_a}$. Consequently, we obtain

$$\begin{aligned} &\langle a_f^+ a_{-f}^+ a_{-f'} a_{f'} \rangle_{H_a} - \langle a_f^+ a_{-f}^+ \rangle_{H_a} \langle a_{-f'} a_{f'} \rangle_{H_a} \\ &\quad = \langle a_f^+ a_{f'} \rangle_{H_a} \langle a_{-f}^+ a_{-f'} \rangle_{H_a} - \langle a_f^+ a_{-f'} \rangle_{H_a} \langle a_{-f}^+ a_{f'} \rangle_{H_a}. \end{aligned} \tag{4.39}$$

But the first term in the right-hand side of (4.39) can be non-zero only if $f' = f$, and the second only if $f' = -f$.

In both these cases, the right-hand side does not exceed unity in absolute magnitude.

Thus, from (4.38), we find

$$\langle (J_\alpha - \bar{C}_\alpha^{(V)})(J_\alpha^+ - \bar{C}_\alpha^{*(V)}) \rangle_{H_a} \leqslant \frac{1}{2V^2} \sum_f |\lambda_\alpha(f)|^2 \leqslant \frac{Q_2}{V}.$$

Hence, on the basis of (4.36) and (4.37), we see that

$$0 \leqslant f_V(H) - f_V\{H(\bar{C}^{(V)})\} \leqslant \frac{2}{V} Q_2 \sum_\alpha g_\alpha. \tag{4.40}$$

On the other hand, taking (4.31) and (4.33) into account, we have

$$\begin{aligned} H(C) &= T_0 - \tfrac{1}{2} \sum_f \{ \Lambda(f) a_f^+ a_{-f}^+ + \Lambda^*(f) a_{-f} a_f \} - 2V \sum_\alpha g_\alpha C_\alpha^* C_\alpha, \\ \Lambda(f) &= -2 \sum_\beta U_\beta^* \mu_\beta(f) - 2 \sum_\alpha g_\alpha C_\alpha^* \lambda_\alpha(f), \end{aligned} \tag{4.41}$$

and, therefore,

$$f_V\{H(C)\} = -2\sum_\alpha g_\alpha C_\alpha^* C_\alpha + \frac{1}{V}\sum_f F(f; C, U), \tag{4.42}$$

where

$$F(f; C, U) = \tfrac{1}{2}\{-E(f) - T(f) - 2\theta \ln (1 + e^{-E(f)/\theta})\},$$

$$E^2(f) = T^2(f) + |A(f)|^2, \qquad T(f) = \frac{p^2}{2m} - \mu. \tag{4.43}$$

By repeating our arguments from Chapter 3, § 2, it is not difficult to show that the difference between the sum and the corresponding integral is bounded by the inequality

$$\left| \frac{1}{V}\sum_f F(f; C, U) - \frac{1}{(2\pi)^3}\int F(f; C, U)\, df \right| \leqslant \varrho_V(\mathcal{M}) \to 0 \quad \text{as} \quad V \to \infty \tag{4.44}$$

for

$$|C_\alpha| \leqslant Q_1, \qquad |U_\beta| \leqslant \mathcal{M}.$$

Here $\varrho_V(\mathcal{M})$ goes to zero as $V \to \infty$ ($\mathcal{M}$ is fixed) uniformly with respect to θ in the interval $(0 < \theta \leqslant \theta_0)$.

Thus,

$$\begin{gathered} |f_V\{H(C)\} - f_\infty\{H(C)\}| \leqslant \varrho_V(\mathcal{M}), \\ |C_\alpha| \leqslant Q_1, \qquad |U_\beta| \leqslant \mathcal{M}, \end{gathered} \tag{4.45}$$

where

$$f_\infty\{H(C)\} = -2\sum_\alpha g_\alpha C_\alpha^* C_\alpha + \frac{1}{(2\pi)^3}\int F(f; C, U)\, df. \tag{4.46}$$

From the above-mentioned arguments it follows also that $f_\infty\{H(C)\}$ is a continuous function of $\ldots C_\alpha \ldots C_\alpha^* \ldots$ and $\ldots U_\beta \ldots U_\beta^* \ldots$, and possesses continuous partial derivatives, of all orders, with respect to these variables.

In addition, we can verify that a point $C = \bar{C}$ exists which realizes the absolute maximum

$$f_\infty\{H(\bar{C})\} = \max_{(C)} f_\infty\{H(C)\},$$

and, for this point,

$$|\bar{C}_\alpha| \leqslant Q_1,$$

and the equations

$$\left.\frac{\partial f_\infty\{H(C)\}}{\partial C_\alpha} = 0, \quad \frac{\partial f_\infty\{H(C)\}}{\partial C_\alpha^*} = 0,\right. \qquad C = \bar{C} \tag{4.47}$$

are satisfied. We note now that, by virtue of (4.45),

$$f_V\{H(\bar{C}^{(V)})\} \geqslant f_V\{H(\bar{C})\} \geqslant f_\infty\{H(\bar{C})\} - \varrho_V(\mathcal{M}),$$
$$f_\infty\{H(\bar{C})\} \geqslant f_\infty\{H(\bar{C}^{(V)})\} \geqslant f_V\{H(\bar{C}^{(V)})\} - \varrho_V(\mathcal{M}),$$

in view of which,

$$|f_V\{H(\bar{C}^{(V)})\} - f_\infty\{H(\bar{C})\}| \leqslant \varrho_V(\mathcal{M}).$$

Hence, taking (4.40) into account, we obtain

$$|f_V(H) - f_\infty\{H(\bar{C})\}| \leqslant \frac{2}{V} Q_2 \sum_\alpha g_\alpha + \varrho_V(\mathcal{M}) \to 0 \quad \text{as} \quad V \to \infty, \tag{4.48}$$

for $|U_\beta| \leqslant \mathcal{M}$.

§ 5. SOLUTION OF THE QUESTION OF UNIQUENESS

We shall make a few more remarks on the equations (4.47).

We shall take the arbitrary complex numbers $z_1, \ldots, z_r$ and the real number t. We consider the second derivative

$$\begin{aligned} \frac{d^2}{dt^2} f_\infty\{H(C+tz)\} &= -4 \sum_\alpha g_\alpha z_\alpha^* z_\alpha \\ &\quad + \frac{1}{(2\pi)^3} \int \left\{ \frac{d^2}{dt^2} F(f; C+tz, U) \right\} df. \end{aligned} \tag{4.49}$$

We note that the function $F(f; C+tz, U)$ is given by the right-hand side of formula (4.43), in which, in the expression for $E(f)$, $\Lambda(f)$ is replaced by

$$\Lambda(f) + t\Lambda_1(f)$$

where

$$\Lambda_1(f) = -2\sum_{\alpha} g_\alpha z_\alpha^* \lambda_\alpha(f).$$

We have, therefore,

$$\frac{d^2F(f; C+tz, U)}{dt^2} = -\frac{1}{8E^3(1+e^{E/\theta})^2}\Big\{2e^{E/\theta}\frac{E}{\theta}$$
$$\times|\Lambda_1(f)(\Lambda^*(f)+t\Lambda_1^*(f))+(\Lambda(f)+t\Lambda_1(f))\Lambda_1^*(f)|^2$$
$$+[4T^2(f)|\Lambda_1(f)|^2-|\Lambda_1(f)(\Lambda^*(f)+t\Lambda_1^*(f))$$
$$-(\Lambda(f)+t\Lambda_1(f))\Lambda_1^*(f)|^2](e^{E/\theta}-1)(e^{E/\theta}+1)\Big\} \leqslant 0,$$

whence, on the basis of (4.49),

$$\frac{d^2f_\infty\{H(C+tz)\}}{dt^2} \leqslant -4\sum_{\alpha} g_\alpha z_\alpha^* z_\alpha \leqslant -4g\sum_{\alpha} z_\alpha^* z_\alpha\,, \tag{4.50}$$

where g is the smallest of the numbers $g_1, \ldots, g_r$. Hence it follows (as in § 1 of this chapter) that the solution of equations (4.47) is unique. These equations have no solution, other than $C = \bar{C} = \bar{C}(U)$, giving an absolute maximum of the function $f_\infty\{H(C)\}$.

As has already been remarked, $\bar{C}(U)$ is defined for any complex $U = (\ldots U_\beta \ldots)$ and is bounded uniformly by the inequalities

$$|\bar{C}_\alpha(U)| \leqslant Q_1\,. \tag{4.51}$$

We shall show now that the $C_\alpha(U)$ are continuous functions and possess continuous partial derivatives with respect to the variables $\ldots U_\beta \ldots U_\beta^* \ldots$.

For this, we shall make use of a well-known theorem about implicit functions. Since, in its usual formulation, this theorem refers to equations with real variables, we now go over from our complex quantities C_α and U_β to real quantities.

We put

$$C_\alpha = x_\alpha + ix_{\alpha+r}\,, \qquad U_\beta = u_\beta + iu_{\beta+l}\,,$$
$$C_\alpha^* = x_\alpha - ix_{\alpha+r}\,, \qquad U_\beta^* = u_\beta - iu_{\beta+l}\,,$$
$$\alpha = 1, \ldots, r, \qquad \beta = 1, \ldots, l.$$

Thus, the expression

$$f_{\infty}\{H(C)\} = \phi(x_1, \ldots, x_{2r};\ u_1, \ldots, u_{2l}) \tag{4.52}$$

will be a real function of the real variables

$$x_1, \ldots, x_{2r}; \qquad u_1, \ldots, u_{2l}.$$

This function will be continuous and will possess continuous partial derivatives of all orders for all real values of the variables x and u.

Putting

$$z_\alpha = y_\alpha + iy_{\alpha+r}, \qquad z_\alpha^* = y_\alpha - iy_{\alpha+r}, \qquad \alpha = 1, 2, \ldots, r,$$

in (4.50), we write this inequality in the form

$$\frac{d^2}{dt^2}\phi(x_1+ty_1, \ldots, x_{2r}+ty_{2r};\ u_1, \ldots, u_{2r}) \leqslant -4g \sum_{1 \leqslant \alpha \leqslant 2r} y_\alpha^2. \tag{4.53}$$

Noting that

$$x_\alpha = \frac{C_\alpha + C_\alpha^*}{2}, \qquad x_{\alpha+r} = -i\frac{(C_\alpha - C_\alpha^*)}{2}, \qquad \alpha = 1, \ldots, r,$$

we have

$$\frac{\partial}{\partial C_\alpha} = \frac{1}{2}\frac{\partial}{\partial x_\alpha} - \frac{i}{2}\frac{\partial}{\partial x_{\alpha+r}}, \qquad \frac{\partial}{\partial C_\alpha^*} = \frac{1}{2}\frac{\partial}{\partial x_\alpha} + \frac{i}{2}\frac{\partial}{\partial x_{\alpha+r}}.$$

Consequently, the equations (4.47) being considered are equivalent to the equations

$$\frac{\partial\phi(x_1, \ldots, x_{2r};\ u_1, \ldots, u_{2l})}{\partial x_\alpha} = 0, \qquad (\alpha = 1, \ldots, 2r). \tag{4.54}$$

Therefore, the equations (4.54) have, for any real $u_1, \ldots u_{2l}$, the unique solution

$$x_\alpha = x_\alpha(u_1, \ldots, u_{2l}), \qquad \alpha = 1, \ldots, 2r.$$

By virtue of (4.51), we have

$$|x_\alpha(u_1, \ldots, u_{2l})| \leqslant Q_1$$

everywhere. Furthermore, taking the inequality (4.50) at $t = 0$, we obtain

$$\sum_{\alpha,\alpha'} \frac{\partial^2 \phi(x, u)}{\partial x_\alpha \, \partial x'_\alpha} y_\alpha y_{\alpha'} \leqslant -4g \sum_\alpha y_\alpha^2 \tag{4.55}$$

for arbitrary real $\ldots x_\alpha \ldots u_\beta \ldots y_\alpha \ldots$. But it follows from the fact that the quadratic form in y in the left-hand side of inequality (4.55) is negative-definite that

$$\operatorname{Det} \left\{ \frac{\partial^2 \phi(x, y)}{\partial x_\alpha \, \partial x'_\alpha} \right\} < 0 \tag{4.56}$$

everywhere.

The conditions for the application of the theorem on implicit functions to the equations (4.54) are thus fulfilled. Thus, we convince ourselves that, everywhere (i.e. for all real u_β), the $x_\alpha(u_1, \ldots, u_{2l})$ are continuous functions and possess continuous partial derivatives with respect to $\ldots u_\beta \ldots$.

Returning to our complex variables C and U, we see that

$$\bar{C}_\alpha = \bar{C}_\alpha(U) \qquad (\alpha = 1, \ldots, r)$$

are indeed continuous functions and possess continuous partial derivatives with respect to $\ldots U_\beta \ldots U_\beta^* \ldots$ in the whole space of the points (U).

To conclude, we summarize the results obtained above in the following theorem.

THEOREM 4.I. *Let the Hamiltonian H be defined by the formulae* (4.30) *and* (4.31) *in which the functions $\lambda_\alpha(f)$ and $\mu_\beta(f)$ satisfy the conditions* (4.24). *Then,*

1. $$|f_V(H) - f_\infty(H)| \leqslant \frac{2}{V} Q_2 \sum_\alpha g_\alpha + \varrho_V(\mathcal{M}),$$

where $\mathcal{M}$ is any number greater than or equal to all the $|U_\beta|$ and where, for fixed $\mathcal{M}$,

$$\varrho_V(\mathcal{M}) \to 0 \quad as \quad V \to \infty$$

uniformly with respect to θ *in the interval* $(0 < \theta \leqslant \theta_0)$.

2.
$$f_\infty(H) = \max_{(C)} f_\infty\{H(C)\}$$

where $H(C)$ *is given by formula* (4.33).

$f_\infty\{H(C)\}$ *is defined by the equality* (4.46) *for all* C *and* U, *and is a continuous function which possesses, everywhere, continuous partial derivatives of all orders with respect to the variables* $\ldots C_\alpha \ldots C_\beta^* \ldots U_\beta \ldots U_\beta^* \ldots$.

3. *The equations*

$$\frac{\partial f_\infty\{H(C)\}}{\partial C_\alpha} = 0, \quad \frac{\partial f_\infty\{H(C)\}}{\partial C_\alpha^*} = 0$$

have, for complex U_β, *a unique solution*

$$C = \bar{C} = \bar{C}(U)$$

This solution realizes the absolute maximum referred to in paragraph 2 of this theorem:

$$f_\infty(H) = \max_{(C)} f_\infty\{H(C)\} = f_\infty\{H(\bar{C}(U))\}.$$

The functions $\bar{C}_\alpha(U)$ *are everywhere continuous and possess continuous partial derivatives with respect to the variables* $\ldots U_\beta \ldots U_\beta^* \ldots$.

§ 6. HAMILTONIANS WITH COUPLING CONSTANTS OF DIFFERENT SIGNS. THE MINIMAX PRINCIPLE

We pass now to the study of a model Hamiltonian with positive and negative interaction terms:

$$H = T_0 + 2V \sum_{(1 \leqslant \alpha \leqslant r)} g_\alpha J_\alpha J_\alpha^+ - 2V \sum_{r+1 \leqslant \alpha \leqslant r+s} g_\alpha J_\alpha J_\alpha^+,$$

$$J_\alpha = \frac{1}{2V} \sum_f \lambda_\alpha(f) a_f^+ a_{-f}^+, \qquad \lambda_\alpha(-f) = -\lambda_\alpha(f), \qquad g_\alpha > 0, \tag{4.57}$$

$$\alpha = 1, \ldots, r+s,$$

$$T_0 = \sum_f \left(\frac{p^2}{2m} - \mu\right) a_f^+ a_f.$$

We shall assume that the functions $\lambda(f) = \lambda(\boldsymbol{p}, \sigma)$ satisfy the conditions[†]

(1) $$|\lambda_\alpha(\boldsymbol{p}, \sigma)| \leq \tilde{Q} = \text{const.}$$

(2) $$\frac{1}{V}\sum_{\boldsymbol{p}} p^2 |\lambda_\alpha(\boldsymbol{p}, \sigma)| \leq Q = \text{const.}$$

(3) The functions $\lambda_\alpha(\boldsymbol{p}, \sigma)$ are continuous everywhere, with the possible exception of a set of points $\boldsymbol{p}$ of measure zero.

We note, first of all, that it follows from conditions 1 and 2 that:

$$\begin{aligned} &\frac{1}{V}\sum_{\boldsymbol{p}} |\lambda_\alpha(\boldsymbol{p}, \sigma)| \leq Q_1 = \text{const.} \\ &\frac{1}{V}\sum_{\boldsymbol{p}} |\lambda_\alpha(\boldsymbol{p}, \sigma)|^2 \leq Q_2 = \text{const.} \end{aligned} \tag{4.58}$$

We now put

$$T = T_0 + 4V \sum_{\alpha=1}^{r} g_\alpha J_\alpha J_\alpha^+ . \tag{4.59}$$

Then,

$$H = T - 2V \sum_{\alpha=1}^{r+s} g_\alpha J_\alpha J_\alpha^+ . \tag{4.60}$$

Starting from the conditions 1 and 2, it is not difficult to verify that, for the Hamiltonian (4.60) with the operator T given by (4.59), the conditions of Theorem 3.I are satisfied.

To verify that the conditions of Theorem 3.II are also fulfilled, we need only to prove that, for any fixed complex S_α $(\alpha = 1, \ldots, r+s)$,

$$f_V\{H_T(S)\} - f_\infty\{H_T(S)\} \to 0 \qquad (V \to \infty), \tag{4.61}$$

where

$$H_T(S) = T_0 + 4V \sum_{\alpha=1}^{r} g_\alpha J_\alpha J_\alpha^+ - 2V \sum_{\alpha=1}^{r+s} g_\alpha \{S_\alpha J_\alpha^+ + S_\alpha^* J_\alpha - S_\alpha S_\alpha^*\}. \tag{4.62}$$

[†] The conditions (1) and (2) will obviously be fulfilled if, for example,

$$|\lambda_\alpha(\boldsymbol{p}, \sigma)| \leq \frac{A}{(p^2+B)^2},$$

where A and B are positive constants.

To demonstrate the validity of (4.61), we note that the expression

$$H_T(S)-\text{const} = H_T(S)-2V\sum_{\alpha=1}^{r+s} g_\alpha S_\alpha S_\alpha^*$$

coincides with the Hamiltonian (4.30) if, in the latter, g_α is replaced by $2g_\alpha$ and U_α by $g_\alpha S_\alpha$, and that the conditions (4.24) for the applicability of Theorem 4.I are fulfilled.

Hence it follows, in particular, that

$$\begin{gathered} |f_V\{H_T(S)\}-f_\infty\{H_T(S)\}| \leqslant \frac{4}{V}Q_2\sum_{\alpha=1}^{r} g_\alpha+\varrho_V(\mathcal{M}), \\ \mathcal{M} \geqslant g_\alpha|S_\alpha|, \quad \alpha = 1, 2, \ldots, r+s, \end{gathered} \tag{4.63}$$

where

$$\varrho_V(\mathcal{M}) \to 0 \qquad (V\to\infty)$$

for any fixed value of $\mathcal{M}$. The asymptotic relation (4.61) is thus proved, and the conditions of applicability of Theorem 3.II are found to be fulfilled.

By making use of this theorem, we can convince ourselves of the existence of points $S=\bar{S}$ at which the function $f_\infty\{H_T(S)\}$ attains its smallest value:

$$f_\infty\{H_T(\bar{S})\} = \min_{(S)} f_\infty\{H_T(S)\}. \tag{4.64}$$

Further, on the basis of (4.63) and paragraph 2 of Theorem 3.II, we see that

$$\begin{gathered} |f_V(H)-f_\infty\{H_T(\bar{S})\}| \leqslant \delta_V = \varepsilon\left(\frac{1}{V}\right)+\delta_V, \\ \delta_V = \frac{4}{V}Q_2\sum_{\alpha=1}^{r} g_\alpha+\varrho(2GQ_1), \end{gathered} \tag{4.65}$$

where G is the largest of the constants $g_1, \ldots, g_r$ and $\varepsilon(1/V)$ is, for the Hamiltonian (4.60), the quantity corresponding to ε in Theorem 3.I.

We emphasize that

$$\delta_V \to 0, \quad \bar{\delta}_V \to 0 \quad \text{as} \quad V\to\infty \tag{4.66}$$

uniformly with respect to θ in the interval $(0 < \theta \leqslant \theta_0)$.

We now turn our attention to the rider to Theorem 3.IV. Since the conditions of Theorem 3.II are fulfilled, we can assert, on the basis of this rider, that

$$0 \leqslant f_V(\Gamma_\tau) - f_V(H) \leqslant \delta_V + \bar{\delta}_V ,$$

and

$$2 \sum_{\alpha=1}^{r+s} g_\alpha \langle (J_\alpha - \bar{S}_\alpha)(J_\alpha^+ - \bar{S}_\alpha^*) \rangle_{\Gamma_\tau} \leqslant \frac{\delta_V + \bar{\delta}_V}{\tau_0} , \tag{4.67}$$

where

$$\begin{gathered} \Gamma_\tau = H + 2V \sum_{\alpha=1}^{r+s} g_\alpha \tau_\alpha (J_\alpha - \bar{S}_\alpha)(J_\alpha^+ - \bar{S}_\alpha^*), \\ 0 < \tau_\alpha < 1, \quad \alpha = 1, \ldots, r+s. \end{gathered} \tag{4.68}$$

τ_0 is the smallest of the numbers τ_α.

Having made these comments, we return to the application of Theorem 4.I in the case of the Hamiltonian $H_T(S)$.

Taking paragraphs 2 and 3 of this theorem into account, we see that

$$f_\infty\{H_T(S)\} = \max_{(C)} f_\infty\{H_T(C, S)\} = f_\infty\big\{H_T(C(S), S)\big\}, \tag{4.69}$$

where

$$\begin{aligned} H_T(C, S) = T_0 + 4V \sum_{\alpha=1}^{r} g_\alpha \{C_\alpha J_\alpha^+ + C_\alpha^* J_\alpha - C_\alpha C_\alpha^*\} \\ -2V \sum_{\alpha=1}^{r+s} g_\alpha \{S_\alpha J_\alpha^+ + S_\alpha^* J_\alpha - S_\alpha S_\alpha^*\}, \end{aligned} \tag{4.70}$$

and where $C = C(S)$ is determined by the equations

$$\frac{\partial}{\partial C_\alpha} f_\infty\{H_T(C, S)\} = 0, \quad \frac{\partial}{\partial C_\alpha^*} f_\infty\{H_T(C, S)\} = 0, \tag{4.71}$$

which, for a given S, have a single unique solution.

It follows from (4.69) that

$$\begin{aligned} f_\infty\{H_T(\bar{S})\} &= f_\infty\big\{H_T(C(\bar{S}), \bar{S})\big\} \\ &= \min_{(S)} f_\infty\big\{H_T(C(S), S)\big\} = \min_{(S)} \max_{(C)} f_\infty\{H_T(C, S)\}, \end{aligned} \tag{4.72}$$

and therefore, by virtue of (4.65), we obtain

$$\lim_{(V\to\infty)} f_V(H) = \min_{(S)} \max_{(C)} f_\infty\{H_T(C,S)\}. \tag{4.73}$$

We have found here an expression for the asymptotic calculation of the free energy by means of the "minimax principle" developed in papers by the author (Bogolyubov, Jr., 1968 f, g, 1969).

We note further that, on the basis of Theorem 4.I, the expression

$$f_\infty\{H_T(C,S)\} \tag{4.74}$$

is a continuous function with continuous partial derivatives with respect to the variables $\ldots C_\alpha \ldots C_\alpha^* \ldots S_\beta \ldots S_\beta^* \ldots$, as is $C_\alpha(S)$ with respect to the variables $\ldots S_\beta \ldots S_\beta^* \ldots$.

Consequently, $f_\infty\{H_T(C(S),S)\}$ is continuous and possesses continuous partial derivatives with respect to the variables $\ldots S_\beta \ldots S_\beta^* \ldots$.

Therefore, if the point $S = \bar{S}$ realizes the absolute minimum of this function, we shall have

$$\frac{\partial f_\infty\{H_T(C(S),S)\}}{\partial S_\beta} = 0, \quad \frac{\partial f_\infty\{H_T(C(S),S)\}}{\partial S_\beta^*} = 0 \tag{4.75}$$

for $S = \bar{S}$. But, in view of the fact that $C = C(S)$ satisfies the equations (4.71), we can write

$$\frac{\partial f_\infty\{H_T(C(S),S)\}}{\partial S_\beta} = \sum_{\alpha=1}^{r} \left\{\frac{\partial f_\infty\{H_T(C,S)\}}{\partial C_\alpha}\right\}_{C=C(S)} \cdot \frac{\partial C_\alpha(S)}{\partial S_\beta} + \left\{\frac{\partial f_\infty\{H_T(C,S)\}}{\partial S_\beta}\right\}_{C=C(S)}.$$

Thus,

$$C_\alpha = \bar{C}_\alpha = C_\alpha(\bar{S}), \qquad S_\beta = \bar{S}_\beta \tag{4.76}$$

is a solution of equations (4.71) and of

$$\frac{\partial f_\infty\{H_T(C,S)\}}{\partial S_\beta} = 0, \quad \frac{\partial f_\infty\{H_T(C,S)\}}{\partial S_\beta^*} = 0. \tag{4.77}$$

But, by definition of the absolute minimum,

$$f_\infty\{H_T(\bar{C},\bar{S})\} \leqslant f_\infty\{H_T(C(S),S)\}.$$

Now let

$$C = \tilde{C}, \quad S = \tilde{S} \tag{4.78}$$

be some other solution of the system of equations (4.71), (4.77). Then it follows immediately from (4.71) that

$$\tilde{C} = C(\tilde{S}).$$

Therefore

$$f_\infty\{H_T(\bar{C}, \bar{S})\} \leqslant f_\infty\{H_T(\tilde{C}, \tilde{S})\}.$$

As can be seen, (4.76) is a solution of equations (4.71), (4.77) that gives the function (4.74) its smallest value on the set of the points (C, S) satisfying the equations (4.71) and (4.77).

Conversely, if (4.78) satisfies (4.71) and (4.77) and gives the function (4.74) its smallest value on this set, then

$$\begin{aligned} f_\infty\big\{H_T(C(\tilde{S}), \tilde{S})\big\} &= f_\infty\{H_T(\tilde{C}, \tilde{S})\} = f_\infty\{H_T(\bar{C}, \bar{S})\} \\ &= \min_{(S)} f_\infty\big\{H_T(C(S), S)\big\}. \end{aligned}$$

Thus, the *minimax principle* for determining $\bar{C}$ and $\bar{S}$, i.e. the determination of $C = C(S)$ from the condition that the function (4.74) be an absolute maximum in the space of all C for given S, followed by the determination of $S = \bar{S}$ from the condition that $f_\infty\big\{H_T(C(S), S)\big\}$ be an absolute minimum in the space of all S, is equivalent to finding a solution of the equations (4.71) and (4.77) that gives the function (4.74) a smaller value than do any other solutions of these equations.

We shall now write the equations (4.71) and (4.77) explicitly.

Starting from the expression (4.70) for $H_T(C, S)$, we find, in the usual way,

$$\begin{aligned} f_\infty\{H_T(C, S)\} &= -4\sum_{\alpha=1}^{r} g_\alpha C_\alpha^* C_\alpha + 2\sum_{\alpha=1}^{r+s} g_\alpha S_\alpha^* S_\alpha \\ &\quad + \frac{1}{2(2\pi)^3}\int \{T(f) - E(f) - 2\theta \ln(1 + e^{-E(f)/\theta})\}\, df, \end{aligned} \tag{4.79}$$

$$T(f) = \frac{p^2}{2m} - \mu, \quad E^2(f) = T^2(f) + |\Delta(f)|^2,$$

$$\Delta(f) = 2\sum_{\alpha=1}^{r+s} g_\alpha \lambda_\alpha(f) S_\alpha^* - 4\sum_{\alpha=1}^{r} g_\alpha \lambda_\alpha(f) C_\alpha^*,$$

and, therefore, the equations under consideration can be represented in the form

$$\begin{aligned} C_\alpha &= \frac{1}{2(2\pi)^3}\int \Delta(f)\,\lambda^*_\alpha(f)\tanh\left(\frac{E(f)}{2\theta}\right)df, \\ C^*_\alpha &= \frac{1}{2(2\pi)^3}\int \Delta^*(f)\,\lambda_\alpha(f)\tanh\left(\frac{E(f)}{2\theta}\right)df, \\ S_\alpha &= \frac{1}{2(2\pi)^3}\int \Delta(f)\,\lambda^*_\alpha(f)\tanh\left(\frac{E(f)}{2\theta}\right)df, \\ S^*_\alpha &= \frac{1}{2(2\pi)^3}\int \Delta^*(f)\,\lambda_\alpha(f)\tanh\left(\frac{E(f)}{2\theta}\right)df. \end{aligned} \tag{4.80}$$

Hence it follows that, for any solution of these equations,

$$C_\alpha = S_\alpha \qquad (\alpha = 1, \ldots, r), \tag{4.81}$$

and

$$f_\infty\{H_T(C, S)\} = f_\infty\{H(C, S)\}, \tag{4.82}$$

where

$$\begin{aligned} H(C, S) &= H(C_1, \ldots, C_r; S_{r+1}, \ldots, S_{r+s}) \\ &= T_0 + 2V\sum_{\alpha=1}^{r} g_\alpha\{C_\alpha J^+_\alpha + C^*_\alpha J_\alpha - C_\alpha C^*_\alpha\} \\ &\quad -2V\sum_{\alpha=r+1}^{r+s} g_\alpha\{S_\alpha J^+_\alpha + S^*_\alpha J_\alpha - S_\alpha S^*_\alpha\}, \end{aligned} \tag{4.83}$$

and

$$\begin{aligned} f_\infty\{H(C, S)\} &= -2\sum_{\alpha=1}^{r} g_\alpha C_\alpha C^*_\alpha + 2\sum_{\alpha=r+1}^{r+s} g_\alpha S_\alpha S^*_\alpha \\ &\quad + \frac{1}{2(2\pi)^3}\int \{T(f) - E(f) - 2\theta\ln(1 + e^{-E(f)/\theta})\}\,df, \end{aligned} \tag{4.84}$$

$$T(f) = \frac{p^2}{2m} - \mu, \quad E(f) = \sqrt{T^2(f) + |\Delta(f)|^2},$$

$$\Delta(f) = 2\sum_{\alpha=r+1}^{r+s} g_\alpha\lambda_\alpha(f)S^*_\alpha - 2\sum_{\alpha=1}^{r} g_\alpha\lambda_\alpha(f)C^*_\alpha .$$

It is clear also that the equations (4.80) are equivalent to the equations:

$$\begin{aligned} &\frac{\partial f_\infty\{H(C, S)\}}{\partial C_\alpha} = 0, \quad \frac{\partial f_\infty\{H(C, S)\}}{\partial C_\alpha^*} = 0, \quad \text{for} \quad \alpha = 1, 2, \ldots, r, \\ &\frac{\partial f_\infty\{H(C, S)\}}{\partial S_\alpha} = 0, \quad \frac{\partial f_\infty\{H(C, S)\}}{\partial S_\alpha^*} = 0, \quad \text{for} \quad \alpha = r+1, \ldots, s, \end{aligned} \tag{4.85}$$

with the additional equalities (4.81). On the basis of (4.82), we can now state that the method formulated earlier for the determination of the quantities $C_1, \ldots, C_r$ and $S_1, \ldots, S_{r+s}$ is equivalent to the following:

(a) The quantities

$$C_1 = \bar{C}_1, \ldots, C_r = \bar{C}_r, \qquad S_{r+1} = \bar{S}_{r+1}, \ldots, S_{r+s} = \bar{S}_{r+s}$$

are also solutions of the system of equations (4.85) that give the function

$$f_\infty\{H(C, S)\} \tag{4.86}$$

its smallest value on the set of all the solutions $\bar{S}_\alpha = \bar{C}_\alpha$ $(\alpha = 1, \ldots, r)$ of these equations.

(b) Repeating the arguments given above for $f_\infty\{H_T(C, S)\}$, we see that such a formulation of the determination of

$$\bar{C}_1, \ldots, \bar{C}_r, \quad \bar{S}_{r+1}, \ldots, \bar{S}_{r+s}$$

is equivalent to the following minimax formulation.

First, we find

$$C = C(S) = C(S_{r+1}, \ldots, S_{r+s})$$

from the condition that the function (4.86) be an absolute maximum in the space of all the points C $(C_1, \ldots, C_r)$ for given S, and then find

$$\bar{S}_{r+1}, \ldots, \bar{S}_{r+s}$$

from the condition that the function $f_\infty\{H(C(S), S)\}$ be an absolute minimum in the space of all the points S $(S_{r+1}, \ldots, S_{r+s})$. It can be seen that $H(C, S)$ is the usual form of the trial Hamiltonian for the Hamiltonian H (4.57). Unlike the form $H_T(C, S)$, which depends on $2r+s$ complex parameters, $H(C, S)$ depends on only $r+s$ complex parameters. Returning to the relation (4.73), we can write it also in the form

$$\lim_{(V\to\infty)} f_V(H) = \min_{(S)} \max_{(C)} f_\infty\{H(C, S)\}. \tag{4.87}$$

To conclude, we turn to the problem of constructing "Hamiltonians with sources" for the determination of quasi-averages.

As in Chapter 3, we shall take

$$\begin{aligned} \Gamma_\tau = H + 2V \sum_{\alpha=1}^{r} g_\alpha \tau_\alpha (J_\alpha - \bar{C}_\alpha)(J_\alpha^+ - \bar{C}_\alpha^*) \\ + 2V \sum_{\alpha=r+1}^{s} g_\alpha \tau_\alpha (J_\alpha - \bar{S}_\alpha)(J_\alpha^+ - \bar{S}_\alpha^*), \end{aligned} \tag{4.88}$$

where τ_α are fixed parameters for which

$$0 < \tau_\alpha < 1, \quad \alpha = 1, \ldots, r+s.$$

It is clear, in view of (4.81), that Γ_τ (4.88) coincides with the Hamiltonian (4.68) and, therefore, the inequalities (4.67) are valid for it. Furthermore, we have

$$\begin{aligned} \Gamma_\tau = H(\bar{C}, \bar{S}) + 2V \sum_{\alpha=1}^{r} g_\alpha (1+\tau_\alpha)(J_\alpha - \bar{C}_\alpha)(J_\alpha^+ - \bar{C}_\alpha^*) \\ - 2V \sum_{\alpha=r+1}^{r+s} g_\alpha (1-\tau_\alpha)(J_\alpha - \bar{S}_\alpha)(J_\alpha^+ - \bar{S}_\alpha^*). \end{aligned} \tag{4.89}$$

Thus, Γ_τ reduces to the form (1.14), (1.15) with

$$\Gamma_a = H(\bar{C}, \bar{S}),$$

$$C_\alpha = \bar{C}_\alpha, \qquad G_\alpha = -2g_\alpha(1+\tau_\alpha), \qquad \alpha = 1, \ldots, r,$$

$$C_\alpha = \bar{S}_\alpha, \qquad G_\alpha = 2g_\alpha(1-\tau_\alpha), \qquad \alpha = r+1, \ldots, r+s.$$

It follows from the inequality (4.67) that

$$\left\langle \sum_{\alpha=1}^{r+s} |G_\alpha| (J_\alpha - C_\alpha)(J_\alpha^+ - C_\alpha^*) \right\rangle_{\Gamma_\tau} \leqslant \varepsilon_V,$$

where

$$\varepsilon_V = 2\frac{\delta_V + \bar{\delta}_V}{\tau_0} \to 0 \qquad (V \to \infty)$$

uniformly with respect to θ in the interval $(0 < \theta \leq \theta_0)$. Clearly, paragraph 3 of the conditions I is fulfilled. The validity of the remaining paragraphs of conditions I and I′ follows trivially from the conditions 1, 2 and 3 of this Section.

We can thus apply the limit theorems of Chapters 1 and 2 to the above case and obtain for the quasi-averages the same results as were obtained in Chapter 3.

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INDEX

Approximating (trial) Hamiltonian 2
Auxiliary Hamiltonian 12

Bardeen–Cooper–Schrieffer (BCS) model 2, 89
Binary correlation averages 26, 29, 56, 69
Bloch–de Dominicis method 69
Bogolyubov transformation 27, 33
Bose–Einstein condensation 23

Canonical u–v transformation 27, 33
Class $\mathscr{L}$ of functions 73
Conditions I 31
Conditions I′ 85
Conservation laws 17, 22, 23
Convergence 70
Coupling constants 15
Crystal lattices 20
Curie point 18

Degeneracy 16

Equations of motion 33

Ferromagnets 17
Field operators 71
Free energy for trial Hamiltonian
  maximization of 13, 138
  minimization of 6, 12, 28, 91
  partial derivatives of 114–116

Generalized convergence 72–76
Green's functions 83

Heisenberg ferromagnet 17
Hölder's inequality 95, 99

Many-time correlation averages 57
Minimax principle 15, 159, 161
Mixed (positive and negative) interaction 154–163
Model Hamiltonian 2, 3, 6, 26, 31
Momentum conservation 22

Negative four-fermion interaction 89–136
Normal ordering 44, 52

Operator field functions 71

Pair sources 26
Partition function 92
Positive four-fermion interaction 137–154

Quasi-averages 5, 11, 16–24, 120
  new method for 124

Repulsive four-fermion interaction 137–154
Riemann integrals 85, 108
Rotation invariance 19

Selection rules 65
Single-time correlation averages 40
Source terms 11, 121
Spectral intensity function 42
Spectral representations 41, 42, 46
Superconductivity 2, 89

Theorem I 55
Theorem II 64
  rider to 71
Theorem III 84
Theorem IV 88
Theorem 3.I 100
Theorem 3.II 105
Theorem 3.III 116
  rider to 117
Theorem 3.IV 12, 127
  rider to 127
Theorem 4.I 153
Time-translation invariance 58
T-products 82
Trial Hamiltonian 2, 27, 31, 90, 91
Two-limit technique 24

Uniform bounds 47, 55, 56, 79, 100, 101, 111–113
u–v transformation 27, 33

Wick's theorem 69